出版前言

2011 年 12 月 11 日是西安交通大学杰出校友钱学森先生的百年诞辰。为缅怀钱学森学长，学习他的科学思想和卓越风范，展示其丰功伟绩和人格魅力，西安交通大学举办了“纪念钱学森诞辰 100 周年”系列活动：作为制片方之一，参与西部电影集团摄制传记故事片《钱学森》；与中央电视台合作，出品纪录片《实验班的故事——沿着钱学森走过的路》；扩建钱学森生平业绩展馆，向校内外开放；举办钱学森科学与教育思想研讨会；出版发行《钱学森力学手稿》、《钱学森年谱（初编）》、《钱学森第六次产业革命思想探微丛书》等。

钱学森先生在美国深造和工作期间留下大量珍贵手稿，这些手稿真实展示了钱学森先生博大精深的学识、开拓求实的精神和严谨奋进的作风，是钱老勇攀科学高峰和严谨治学的集中体现。这里，我们将部分原稿整理汇集成册，出版《钱学森力学手稿》，作为钱老百年诞辰的献礼。

《钱学森力学手稿》共 10 卷，包含两部分内容。第一部分是草稿，包括扁壳、球壳和圆柱壳屈曲分析的公式推导和数值演算。在研究圆柱壳轴压屈曲问题时，为了求得圆柱壳体的临界压力，在有关的五百多页草稿中，对多达二十多种可能的屈曲模

态逐一进行公式推演和数值计算，最终才找到满意的并在论文中采用的屈曲模态。仔细观察草稿中的数据列表，每个数字有效位数都长达八位，在手摇机械式计算机作为主要计算工具的年代，这串串数字凝聚着多少现今难以想象的艰辛劳动。

第二部分是手稿，以航空航天工程为核心，涵盖空气动力学、固体力学、火箭技术、工程控制论和物理力学等领域的部分学术论文手稿、打印稿和讲义。

《钱学森力学手稿》是在西安交通大学校领导的大力支持下，由西安交通大学航天航空学院沈亚鹏教授整理完成。图书出版过程中得到了西安交通大学党委宣传部、校友关系发展部、图书馆、航天航空学院等的积极协助，在此深表感谢。

Preliminary Calculation of Circular Cylinder (Ⅲ)

New Calculation

for the circular region,

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta\right\}$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta - f_1\left[1-\left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1-\left(\frac{r}{a}\right)^2\right]^6\right\}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{R}\left\{-\sin^2\theta + 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 6f_2\left[1-\left(\frac{r}{a}\right)^2\right]^5\right\}$$

$$\frac{1}{r}\frac{\partial w_0}{\partial r} = \frac{1}{R}\left\{-\sin^2\theta\right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{R}\left\{-\sin^2\theta + 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 6f_2\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{-\sin^2\theta\right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{-\cos 2\theta\right\}$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$=\frac{1}{R^2}\Big[(1-\cos 2\theta)\Big\{2f_1\Big[1-2\Big(\frac{r}{a}\Big)^2\Big]+6f_2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^4\Big[1-6\Big(\frac{r}{a}\Big)^2\Big]\Big\}$$

$$-\Big\{4f_1^2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]\Big[1-3\Big(\frac{r}{a}\Big)^2\Big]+24f_1f_2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^5\Big[1-7\Big(\frac{r}{a}\Big)^2\Big]+36f_2^2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^9\Big[1-11\Big(\frac{r}{a}\Big)^2\Big]\Big\}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\}=\frac{1}{R^2}\cos 2\theta\Big\{2f_1\Big[1-3\Big(\frac{r}{a}\Big)^2\Big]+6f_2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^4\Big[1-11\Big(\frac{r}{a}\Big)^2\Big]\Big\}$$

The terms for the particular integral is then, <u>multiplied by R^2</u>

$$2f_1\Big[1-2\Big(\frac{r}{a}\Big)^2\Big]+6f_2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^4\Big[1-6\Big(\frac{r}{a}\Big)^2\Big]-4f_1^2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]\Big[1-3\Big(\frac{r}{a}\Big)^2\Big]$$

$$-24f_1f_2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^5\Big[1-7\Big(\frac{r}{a}\Big)^2\Big]-36f_2^2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^9\Big[1-11\Big(\frac{r}{a}\Big)^2\Big]$$

$$+\cos 2\theta\Big\{-2f_1\Big(\frac{r}{a}\Big)^2-30f_2\Big[1-\Big(\frac{r}{a}\Big)^2\Big]^4\Big(\frac{r}{a}\Big)^2\Big\}$$

433

1	$\left(\frac{r}{a}\right)^{2}$	$\left(\frac{r}{a}\right)^{4}$	$\left(\frac{r}{a}\right)^{6}$	$\left(\frac{r}{a}\right)^{8}$	$\left(\frac{r}{a}\right)^{10}$	$\left(\frac{r}{a}\right)^{12}$	$\left(\frac{r}{a}\right)^{14}$	$\left(\frac{r}{a}\right)^{16}$	$\left(\frac{r}{a}\right)^{18}$	$\left(\frac{r}{a}\right)^{20}$
$+2f_1$	$-4f_1$									
$+6f_2$	$-24f_2$ $-36f_2$	$+36f_2$ $+144f_2$	$-24f_2$ $-216f_2$	$+6f_2$ $+144f_2$	$-36f_2$					
$-4f_1^2$	$+16f_1^2$	$-12f_1^2$								
$-48f_1f_2$	$+240f_1f_2$ $+336f_1f_2$	$-480f_1f_2$ $-1680f_1f_2$	$+480f_1f_2$ $+3360f_1f_2$	$-240f_1f_2$ $-3360f_1f_2$	$+48f_1f_2$ $+1680f_1f_2$	$-336f_1f_2$	$\times$	$\frac{1}{2}$		
$-36f_2^2$	$+324f_2^2$ $+396f_2^2$	$-1296f_2^2$ $-3564f_2^2$	$+3024f_2^2$ $+14256f_2^2$	$-4536f_2^2$ $-33264f_2^2$	$+4536f_2^2$ $+49896f_2^2$	$-3024f_2^2$ $-49896f_2^2$	$+1296f_2^2$ $+33264f_2^2$	$-324f_2^2$ $-14256f_2^2$	$+36f_2^2$ $+3564f_2^2$	$-396f_2^2$
$\frac{1}{16\cdot 4}$	$\frac{1}{36\cdot 16}$	$\frac{1}{64\cdot 36}$	$\frac{1}{100\cdot 64}$	$\frac{1}{144\cdot 100}$	$\frac{1}{196\cdot 144}$	$\frac{1}{256\cdot 196}$	$\frac{1}{324\cdot 256}$	$\frac{1}{400\cdot 324}$	$\frac{1}{484\cdot 400}$	$\frac{1}{576\cdot 484}$
	$-2f_1$									
	$-30f_2$	$+120f_2$	$-180f_2$	$+120f_2$	$-30f_2$					
	$\frac{1}{32\cdot 12}$	$\frac{1}{60\cdot 32}$	$\frac{1}{96\cdot 60}$	$\frac{1}{140\cdot 96}$	$\frac{1}{192\cdot 140}$					

$$\frac{\Phi}{R^2} = E\left(\frac{a}{R}\right)^4\Big[\frac{1}{4}q_0\left(\frac{r}{a}\right)^2 + \frac{1}{32}(f_1+3f_2-2f_1^2-12f_1f_2-18f_2^2)\left(\frac{r}{a}\right)^4 + \frac{1}{144}(-f_1-15f_2+4f_1^2+72f_1f_2+180f_2^2)\left(\frac{r}{a}\right)^6$$

$$+\frac{1}{192}(15f_2-f_1^2-90f_1f_2-405f_2^2)\left(\frac{r}{a}\right)^8 + \frac{1}{400}(-15f_2+120f_1f_2+1080f_2^2)\left(\frac{r}{a}\right)^{10}$$

$$+\frac{1}{9.6}(f_2-12f_1f_2-1252f_2^2)\left(\frac{r}{a}\right)^{12} + \frac{1}{784}(-f_2+24f_1f_2+1512f_2^2)\left(\frac{r}{a}\right)^{14}$$

$$+\frac{1}{896}(-3f_1f_2-945f_2^2)\left(\frac{r}{a}\right)^{16} + \frac{5}{12}f_2^2\left(\frac{r}{a}\right)^{18} - \frac{9}{80}f_2^2\left(\frac{r}{a}\right)^{20} + \frac{9}{484}f_2^2\left(\frac{r}{a}\right)^{22} - \frac{1}{704}f_2^2\left(\frac{r}{a}\right)^{24}$$

$$+\cos 2\theta\Big\{-\frac{1}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^6 + \frac{1}{16}f_2\left(\frac{r}{a}\right)^8 - \frac{1}{32}f_2\left(\frac{r}{a}\right)^{10} + \frac{1}{112}f_2\left(\frac{r}{a}\right)^{12} - \frac{1}{896}f_2\left(\frac{r}{a}\right)^{14} + p_2\left(\frac{r}{a}\right)^2 + h_2\left(\frac{r}{a}\right)^4\Big\}\Big]$$

$$\frac{1}{r}\frac{\partial\Phi}{\partial r} = E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}q_0 + \frac{1}{8}A\left(\frac{r}{a}\right)^2 + \frac{1}{24}B\left(\frac{r}{a}\right)^4 + \frac{1}{24}C\left(\frac{r}{a}\right)^6 + \frac{1}{40}D\left(\frac{r}{a}\right)^8 + \frac{1}{8}E\left(\frac{r}{a}\right)^{10} + \frac{1}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+\frac{1}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+\cos 2\theta\Big\{-\frac{6}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{8}{16}f_2\left(\frac{r}{a}\right)^6 - \frac{10}{32}f_2\left(\frac{r}{a}\right)^8 + \frac{12}{112}f_2\left(\frac{r}{a}\right)^{10} - \frac{14}{896}f_2\left(\frac{r}{a}\right)^{12} + 2p_2 + 4h_2\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ \frac{4}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 - \frac{4}{16}f_2\left(\frac{r}{a}\right)^6 + \frac{4}{32}f_2\left(\frac{r}{a}\right)^8 - \frac{4}{112}f_2\left(\frac{r}{a}\right)^{10} + \frac{4}{896}f_2\left(\frac{r}{a}\right)^{12} - 4p_2 - 4r_2\left(\frac{r}{a}\right)^2 \right\}$$

$$\widehat{rr} = E\left(\frac{a}{R}\right)^2 \Bigg[\frac{1}{2}q_0 + \frac{1}{8}A\left(\frac{r}{a}\right)^2 + \frac{1}{24}B\left(\frac{r}{a}\right)^4 + \frac{1}{24}C\left(\frac{r}{a}\right)^6 + \frac{1}{40}D\left(\frac{r}{a}\right)^8 + \frac{1}{8}E\left(\frac{r}{a}\right)^{10} + \frac{1}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{1}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta \left\{ -\frac{2}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{4}{16}f_2\left(\frac{r}{a}\right)^6 - \frac{6}{32}f_2\left(\frac{r}{a}\right)^8 + \frac{8}{112}f_2\left(\frac{r}{a}\right)^{10} - \frac{10}{896}f_2\left(\frac{r}{a}\right)^{12} - 2p_2 \right\} \Bigg]$$

$$\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Bigg[\frac{1}{2}q_0 + \frac{3}{8}A\left(\frac{r}{a}\right)^2 + \frac{5}{24}B\left(\frac{r}{a}\right)^4 + \frac{7}{24}C\left(\frac{r}{a}\right)^6 + \frac{9}{40}D\left(\frac{r}{a}\right)^8 + \frac{11}{8}E\left(\frac{r}{a}\right)^{10} + \frac{13}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{15}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15\times 17}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9\times 19}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9\times 21}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3\times 23}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta \left\{ -\frac{30}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{56}{16}f_2\left(\frac{r}{a}\right)^6 - \frac{90}{32}f_2\left(\frac{r}{a}\right)^8 + \frac{132}{112}f_2\left(\frac{r}{a}\right)^{10} - \frac{182}{896}f_2\left(\frac{r}{a}\right)^{12} + 2p_2 + 12r_2\left(\frac{r}{a}\right)^2 \right\} \Bigg]$$

$$\widehat{r\theta} = E\left(\frac{a}{R}\right)^2 \sin 2\theta \left\{ -\frac{10}{192}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{14}{16}f_2\left(\frac{r}{a}\right)^6 - \frac{18}{32}f_2\left(\frac{r}{a}\right)^8 + \frac{22}{112}f_2\left(\frac{r}{a}\right)^{10} - \frac{26}{896}f_2\left(\frac{r}{a}\right)^{12} + 2p_2 + 6r_2\left(\frac{r}{a}\right)^2 \right\}$$

$$\widehat{rr} - \nu\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}(1-\nu)q_0 + \frac{(1-3\nu)}{8}A\left(\frac{r}{a}\right)^2 + \frac{(1-5\nu)}{24}B\left(\frac{r}{a}\right)^4 + \frac{(1-7\nu)}{24}C\left(\frac{r}{a}\right)^6 + \frac{(1-9\nu)}{40}D\left(\frac{r}{a}\right)^8 + \frac{(1-11\nu)}{18}E\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{(1-13\nu)}{56}F\left(\frac{r}{a}\right)^{12}$$

$$+ \frac{(1-15\nu)}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15(1-17\nu)}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9(1-19\nu)}{4}I\left(\frac{r}{a}\right)^{18} + \frac{9(1-21\nu)}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3(1-23\nu)}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta\Big\{-\frac{(1-15\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^4 + \frac{(1-14\nu)}{4}f_2\left(\frac{r}{a}\right)^6 - \frac{(3-45\nu)}{16}f_2\left(\frac{r}{a}\right)^8 + \frac{(2-33\nu)}{28}f_2\left(\frac{r}{a}\right)^{10} - \frac{(5-91\nu)}{448}f_2\left(\frac{r}{a}\right)^{12}$$

$$- 2(1+\nu)p_2 - 12\nu R_2\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2 = \left(\frac{a}{R}\right)^2\Big[-f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2 - 3f_2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left(\frac{r}{a}\right)^2 + 2f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(\frac{r}{a}\right)^2 + 12f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^6\left(\frac{r}{a}\right)^2$$

$$+ 18f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^{10}\left(\frac{r}{a}\right)^2 + \cos 2\theta\Big\{f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2 + 3f_2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$= \left(\frac{a}{R}\right)^2\Big[-A\left(\frac{r}{a}\right)^2 - B\left(\frac{r}{a}\right)^4 - 2C\left(\frac{r}{a}\right)^6 - 2D\left(\frac{r}{a}\right)^8 - 15E\left(\frac{r}{a}\right)^{10} - 3F\left(\frac{r}{a}\right)^{12} - 4G\left(\frac{r}{a}\right)^{14} - 2160H\left(\frac{r}{a}\right)^{16}$$

$$+ 810\,I\left(\frac{r}{a}\right)^{18} - 180J\left(\frac{r}{a}\right)^{20} + 18K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta\Big\{(f_1+3f_2)\left(\frac{r}{a}\right)^2 - (f_1+15f_2)\left(\frac{r}{a}\right)^4 + 30f_2\left(\frac{r}{a}\right)^6 - 30f_2\left(\frac{r}{a}\right)^8 + 15f_2\left(\frac{r}{a}\right)^{10} - 3f_2\left(\frac{r}{a}\right)^{12}\Big\}\Big]$$

$(\frac{r}{a})^2$	$(\frac{r}{a})^4$	$(\frac{r}{a})^6$	$(\frac{r}{a})^8$	$(\frac{r}{a})^{10}$	$(\frac{r}{a})^{12}$	$(\frac{r}{a})^{14}$	$(\frac{r}{a})^{16}$	$(\frac{r}{a})^{18}$	$(\frac{r}{a})^{20}$	$(\frac{r}{a})^{22}$
$-f_1$	$+f_1$									
$-3f_2$	$+15f_2$	$-30f_2$	$+30f_2$	$-15f_2$	$+3f_2$					
$+2f_1^2$	$-4f_1^2$	$+2f_1^2$								
$+12f_1f_2$	$-72f_1f_2$	$+180f_1f_2$	$-240f_1f_2$	$+180f_1f_2$	$-72f_1f_2$	$+12f_1f_2$				
$+18f_2^2$	$-180f_2^2$	$+810f_2^2$	$-2160f_2^2$	$+3780f_2^2$	$-4536f_1f_2$	$+3780f_2^2$	$-2160f_2^2$	$+810f_2^2$	$-180f_2^2$	$+18f_2^2$

$+f_1$	$-f_1$				
$+3f_2$	$-15f_2$	$+30f_2$	$-30f_2$	$+15f_2$	$-3f_2$

$$\frac{\partial u}{\partial r} = \left(\frac{q}{R}\right)^2 \Big[\frac{1}{2}(1-\nu)q_0 + \frac{3(3-\nu)}{8}A\left(\frac{r}{a}\right)^2 + \frac{5(5-\nu)}{24}B\left(\frac{r}{a}\right)^4 + \frac{7(7-\nu)}{24}C\left(\frac{r}{a}\right)^6 + \frac{9(9-\nu)}{40}D\left(\frac{r}{a}\right)^8$$

$$+ \frac{11(11-\nu)}{8}E\left(\frac{r}{a}\right)^{10} + \frac{13(13-\nu)}{56}F\left(\frac{r}{a}\right)^{12} + \frac{15(15-\nu)}{56}G\left(\frac{r}{a}\right)^{14} + \frac{15\times17(17-\nu)}{2}H\left(\frac{r}{a}\right)^{16} - \frac{9\times19(19-\nu)}{4}I\left(\frac{r}{a}\right)^{18}$$

$$+ \frac{9\times21(21-\nu)}{22}J\left(\frac{r}{a}\right)^{20} - \frac{3\times23(23-\nu)}{88}K\left(\frac{r}{a}\right)^{22}$$

$$+ \cos 2\theta \Big\{ -(f_1+3f_2)\left(\frac{r}{a}\right)^2 + \frac{(95+15\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^4 - \frac{(119+14\nu)}{4}f_2\left(\frac{r}{a}\right)^6 + \frac{(477+45\nu)}{16}f_2\left(\frac{r}{a}\right)^8$$

$$- \frac{(418+33\nu)}{28}f_2\left(\frac{r}{a}\right)^{10} + \frac{(1339+91\nu)}{448}f_2\left(\frac{r}{a}\right)^{12} - 7(1+\nu)p_2 - 12\nu R_2\left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\frac{u}{R} = \left(\frac{q}{R}\right)^3 \Big[\frac{1}{2}(1-\nu)q_0\left(\frac{r}{a}\right) + \frac{(3-\nu)}{8}A\left(\frac{r}{a}\right)^3 + \frac{(5-\nu)}{24}B\left(\frac{r}{a}\right)^5 + \frac{(7-\nu)}{24}C\left(\frac{r}{a}\right)^7 + \frac{(9-\nu)}{40}D\left(\frac{r}{a}\right)^9$$

$$+ \frac{(11-\nu)}{8}E\left(\frac{r}{a}\right)^{11} + \frac{(13-\nu)}{56}F\left(\frac{r}{a}\right)^{13} + \frac{(15-\nu)}{56}G\left(\frac{r}{a}\right)^{15} + \frac{15(17-\nu)}{2}H\left(\frac{r}{a}\right)^{17} - \frac{9(19-\nu)}{4}I\left(\frac{r}{a}\right)^{19}$$

$$+ \frac{9(21-\nu)}{22}J\left(\frac{r}{a}\right)^{21} - \frac{3(23-\nu)}{88}K\left(\frac{r}{a}\right)^{23}$$

$$+ \cos 2\theta \Big\{ -\frac{1}{3}(f_1+3f_2)\left(\frac{r}{a}\right)^3 + \frac{(19+3\nu)}{96}(f_1+15f_2)\left(\frac{r}{a}\right)^5 - \frac{(17+2\nu)}{4}f_2\left(\frac{r}{a}\right)^7 + \frac{(53+5\nu)}{16}f_2\left(\frac{r}{a}\right)^9$$

$$- \frac{(38+3\nu)}{28}f_2\left(\frac{r}{a}\right)^{11} + \frac{(103+7\nu)}{448}f_2\left(\frac{r}{a}\right)^{13} - 7(1+\nu)p_2\left(\frac{r}{a}\right) - 4\nu R_2\left(\frac{r}{a}\right)^3 \Big\} \Big]$$

The non-uniform part of $\widehat{\theta\theta} - \nu\,\widehat{rr}$ is

$$E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ -\frac{(15-\nu)}{96}(f_1 + 15f_2)\left(\frac{r}{a}\right)^4 + \frac{(14-\nu)}{4} f_2 \left(\frac{r}{a}\right)^6 - \frac{(45-3\nu)}{16} f_2 \left(\frac{r}{a}\right)^8 + \frac{(33-2\nu)}{28} f_2 \left(\frac{r}{a}\right)^{10} - \frac{(91-5\nu)}{488} f_2 \left(\frac{r}{a}\right)^{12} \right.$$

$$\left. + 2(1+\nu)p_2 + 12 h_2 \left(\frac{r}{a}\right)^2 \right\}]$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ \frac{1}{3}(f_1 + 3f_2)\left(\frac{r}{a}\right)^2 - \frac{(34+2\nu)}{96}(f_1 + 15f_2)\left(\frac{r}{a}\right)^4 + \frac{(31+\nu)}{4} f_2 \left(\frac{r}{a}\right)^6 - \frac{(98+2\nu)}{16} f_2 \left(\frac{r}{a}\right)^8 \right.$$

$$\left. + \frac{(71+\nu)}{28} f_2 \left(\frac{r}{a}\right)^{10} - \frac{(194+2\nu)}{448} f_2 \left(\frac{r}{a}\right)^{12} + 4(1+\nu)p_2 + 4(3+\nu)\left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \sin 2\theta \left\{ \frac{1}{6}(f_1 + 3f_2)\left(\frac{r}{a}\right)^3 - \frac{(17+\nu)}{96}(f_1 + 15f_2)\left(\frac{r}{a}\right)^5 + \frac{(31+\nu)}{8} f_2 \left(\frac{r}{a}\right)^7 - \frac{(49+\nu)}{16} f_2 \left(\frac{r}{a}\right)^9 \right.$$

$$\left. + \frac{(71+\nu)}{56} f_2 \left(\frac{r}{a}\right)^{11} - \frac{(97+\nu)}{448} f_2 \left(\frac{r}{a}\right)^{13} + 2(1+\nu)p_2 + 2(3+\nu) h_2 \left(\frac{r}{a}\right)^3 \right\}$$

---

now the non-uniform parts

$$\widehat{rr}_a = \cos 2\theta \left\{ -\frac{1}{96} f_1 - \frac{15}{448} f_2 - 2p_2 \right\} \text{———} \left\{ \frac{1}{2} - 6q_2 - 4s_2 \right\}$$

$$\widehat{\theta\theta}_a = \cos 2\theta \left\{ -\frac{5}{32} f_1 - \frac{305}{448} f_2 + 2p_2 + 12r_2 \right\} \text{———} \left\{ 6q_2 - \frac{1}{2} \right\}$$

$$\widehat{r\theta}_a = \sin 2\theta \left\{ -\frac{5}{96} f_1 - \frac{135}{448} f_2 + 2p_2 + 6r_2 \right\} \text{- - - -} \left\{ -\frac{1}{2} - 6q_2 - 2s_2 \right\}$$

$$\left(\frac{u}{R}\right)_a = \cos 2\theta \left\{ -\frac{(13-3\nu)}{96} f_1 - \frac{43-85\nu}{448} f_2 - 2(1+\nu)p_2 - 4\nu r_2 \right\} - \left\{ \frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2 \right\}$$

$$\left(\frac{v}{R}\right)_a = \sin 2\theta \left\{ -\frac{(1+\nu)}{96} f_1 - \frac{131+35\nu}{448} f_2 + 2(1+\nu)p_2 + 2(3+\nu)r_2 \right\} - \left\{ 2(1+\nu)q_2 - \frac{1}{2}(1+\nu) \right\}$$

$$\frac{.017.5}{1164.8} \qquad \frac{141.5}{1164.8}$$

$$q_2 + 0.66666667 s_2 - 0.08333333 = \eta\{+0.33333333 p_2 + 0 + 0.00173611 f_1 + 0.00558036 f_2\}$$ **442**

$$q_2 + 0 - 0.08333333 = \eta\{+0.33333333 p_2 + 2 r_2 - 0.02604167 f_1 - 0.11346726 f_2\}$$

$$q_2 + 0.3333333 s_2 + 0.0833333 = \eta\{-0.3333333 p_2 - r_2 + 0.00868056 f_1 + 0.05022321 f_2\}$$

$$q_2 + 1.53846154 s_2 + 0.25000 = \eta\{- p_2 - 0.46153846 r_2 - 0.04847756 f_1 - 0.01502404 f_2\}$$

$$q_2 + 0 - 0.25000 = \eta\{+ p_2 + 2.53846154 r_2 - 0.00520833 f_1 - 0.12148008 f_2\}$$

$$0.66666667 s_2 + 0 = \eta\{0 - 2 r_2 + 0.02777778 f_1 + 0.11904762 f_2\}$$

$$0.3333333 s_2 + 0.1666667 = \eta\{-0.6666667 p_2 - 3 r_2 + 0.03472222 f_1 + 0.16369047 f_2\}$$

$$1.20512821 s_2 + 0.1666667 = \eta\{-0.6666667 p_2 + 0.53846154 r_2 - 0.05715812 f_1 - 0.06524735 f_2\}$$

$$1.53846154 s_2 + 0.50000 = \eta\{-2 p_2 - 3 r_2 - 0.04326923 f_1 + 0.10645604 f_2\}$$

$$s_2 + 0 = \eta\{0 - 3 r_2 + 0.04166667 f_1 + 0.17857143 f_2\}$$

$$s_2 + 0.50000 = \eta\{-2 p_2 - 9 r_2 + 0.10416667 f_1 + 0.49107141 f_2\}$$

$$s_2 + 0.13829787 = \eta\{-0.55319149 p_2 + 0.44680851 r_2 - 0.04742908 f_1 - 0.05414133 f_2\}$$

$$s_2 + 0.325000 = \eta\{-1.3 p_2 - 1.95 r_2 - 0.028125000 f_1 + 0.06919643 f_2\}$$

$$0.50000000 = \eta\{-2 p_2 - 6 r_2 + 0.06250000 f_1 + 0.31250000 f_2\}$$

$$0.36170213 = \eta\{-1.44680851 p_2 - 9.44680851 r_2 + 0.15159575 f_1 + 0.54521274 f_2\}$$

$$0.18670213 = \eta\{-0.74680851 p_2 - 2.39680851 r_2 + 0.01930408 f_1 + 0.12333771 f_2\}$$

$$0.25000000 = \eta\left\{-p_2 - 3r_2 + 0.03125000f_1 + 0.15625000f_2\right\}$$

$$0.25000000 = \eta\left\{-p_2 - 6.52941176\,r_2 + 0.10477942f_1 + 0.37683822f_2\right\}$$

$$0.2500000 = \eta\left\{-p_2 - 3.20940171\,r_2 + 0.02584877f_1 + 0.16515313f_2\right\}$$

$$3.52941176\,r_2 = 0.07352942f_1 + 0.22058822f_2$$

$$3.32001005\,r_2 = 0.07893065f_1 + 0.21168509f_2$$

$$3r_2 = 0.06250000f_1 + 0.18750000f_2$$

$$3r_2 = 0.07132266f_1 + 0.19128113f_2$$

$$0 = 0.00882266f_1 + 0.00378113f_2$$

New Calculation

For the circular region,

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta \right\}$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ 1 - \left(\frac{r}{a}\right)^2 \sin^2\theta - f_1\left[1-\left(\frac{r}{a}\right)^2\right]^2 - f_2\left[1-\left(\frac{r}{a}\right)^2\right]^4 \right\}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{R}\left\{ -\sin^2\theta + 2f_1\left[1-\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3 \right\}$$

$$\frac{1}{r}\frac{\partial w_0}{\partial r} = \frac{1}{R}\left\{ -\sin^2\theta \right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{R}\left\{ -\sin^2\theta + 2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-7\left(\frac{r}{a}\right)^2\right] \right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{R}\left\{ -\sin^2\theta \right\}.$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial \theta^2} = \frac{1}{R}\left\{ -\cos 2\theta \right\}$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

$$= \frac{1}{R^2}\left[(1-\cos 2\theta)\left\{2f_1\left[1-2\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-4\left(\frac{r}{a}\right)^2\right]\right\}\right.$$

$$\left. - \left\{4f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]\left[1-3\left(\frac{r}{a}\right)^2\right] + 8f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left[2-10\left(\frac{r}{a}\right)^2\right] + 16f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left[1-7\left(\frac{r}{a}\right)^2\right]\right\}\right]$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial \theta^2} - \frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial \theta^2}\right\} = \frac{1}{R^2}\cos 2\theta\left\{2f_1\left[1-3\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-7\left(\frac{r}{a}\right)^2\right]\right\}$$

The terms for the particular integral is then, <u>multiplied by R^2</u>

$$2f_1\left[1-2\left(\frac{r}{a}\right)^2\right] + 4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left[1-4\left(\frac{r}{a}\right)^2\right] - 4f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]\left[1-3\left(\frac{r}{a}\right)^2\right] - 16f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left[1-5\left(\frac{r}{a}\right)^2\right]$$

$$-16f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^5\left[1-7\left(\frac{r}{a}\right)^2\right]$$

$$+\cos 2\theta\left\{-2f_1\left(\frac{r}{a}\right)^2 - 12f_2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(\frac{r}{a}\right)^2\right\}$$

1	$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^4$	$\left(\frac{r}{a}\right)^6$	$\left(\frac{r}{a}\right)^8$	$\left(\frac{r}{a}\right)^{10}$	$\left(\frac{r}{a}\right)^{12}$
$+2f_1$	$-4f_1$					
$+4f_2$	$-8f_2$	$+4f_2$				
	$-16f_2$	$+32f_2$	$-16f_2$			
$-4f_1^2$	$+16f_1^2$	$-12f_1^2$				
$-16f_1f_2$	$+48f_1f_2$	$-48f_1f_2$	$+16f_1f_2$			
	$+80f_1f_2$	$-240f_1f_2$	$+240f_1f_2$	$+80f_1f_2$		
$-16f_2^2$	$+80f_2^2$	$-160f_2^2$	$+160f_2^2$	$-80f_2^2$	$+16f_2^2$	
	$+112f_2^2$	$-560f_2^2$	$+1120f_2^2$	$-1120f_2^2$	$+560f_2^2$	$-112f_2^2$
$\frac{1}{16\cdot 4}$	$\frac{1}{36\cdot 16}$	$\frac{1}{64\cdot 36}$	$\frac{1}{100\cdot 64}$	$\frac{1}{144\cdot 100}$	$\frac{1}{196\cdot 144}$	$\frac{1}{256\cdot 196}$

	$\left(\frac{r}{a}\right)^2$	$\left(\frac{r}{a}\right)^4$	$\left(\frac{r}{a}\right)^6$
$\cos 2\theta$	$-2f_1$		
	$-12f_2$	$+24f_2$	$-12f_2$
	$\frac{1}{32\cdot 12}$	$\frac{1}{60\cdot 32}$	$\frac{1}{96\cdot 60}$

$$\frac{\Phi}{R^2} = E\left(\frac{a}{R}\right)^4 \Big[\frac{1}{4} p_0 \left(\frac{r}{a}\right)^2 + \frac{1}{32}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^4 + \frac{1}{144}\left(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2\right)\left(\frac{r}{a}\right)^6$$

$$+ \frac{1}{192}\left(3f_2 - f_1^2 - 24f_1f_2 - 60f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{400}\left(-f_2 + 16f_1f_2 + 80f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{1}{180}\left(f_1f_2 - 15f_2^2\right)\left(\frac{r}{a}\right)^{12} + \frac{1}{49} f_2^2 \left(\frac{r}{a}\right)^{14} - \frac{1}{448} f_2^2 \left(\frac{r}{a}\right)^{16}$$

$$+ \cos 2\theta \left\{ - \frac{1}{192}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{80} f_2 \left(\frac{r}{a}\right)^8 - \frac{1}{480} f_2 \left(\frac{r}{a}\right)^{10} + p_2 \left(\frac{r}{a}\right)^2 + r_2 \left(\frac{r}{a}\right)^4 \right\} \Big]$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} p_0 + \frac{1}{8}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^2 + \frac{1}{24}\left(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{1}{24}\left(3f_2 - f_1^2 - 24f_1f_2 - 60f_2^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{40}\left(-f_2 + 16f_1f_2 + 80f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{15}\left(f_1f_2 - 15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{2}{7} f_2^2 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_2^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \left\{ - \frac{6}{192}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 + \frac{8}{80} f_2 \left(\frac{r}{a}\right)^6 - \frac{10}{480} f_2 \left(\frac{r}{a}\right)^8 \right.$$

$$\left. + 2p_2 + 4r_2 \left(\frac{r}{a}\right)^2 \right\} \Big]$$

$$\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ + \frac{4}{192}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 - \frac{4}{80} f_2 \left(\frac{r}{a}\right)^6 + \frac{4}{480} f_2 \left(\frac{r}{a}\right)^{10} - 4p_2 - 4r_2 \left(\frac{r}{a}\right)^4 \right\}$$

$$\widehat{rr} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} q_0 + \frac{1}{8}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^2 + \frac{1}{24}\left(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{1}{24}\left(3f_2 - f_1^2 - 24f_1f_2 - 60f_2^2\right)\left(\frac{r}{a}\right)^6 + \frac{1}{40}\left(-f_2 + 16f_1f_2 + 80f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{1}{15}\left(f_1f_2 - 15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{2}{7} f_2^2 \left(\frac{r}{a}\right)^{12} - \frac{1}{28} f_2^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \left\{ -\frac{1}{96}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 + \frac{1}{20} f_2 \left(\frac{r}{a}\right)^6 - \frac{1}{80} f_2 \left(\frac{r}{a}\right)^{10} - 2p_2 \right\}$$

$$\widehat{\theta\theta} = E\left(\frac{a}{R}\right)^2 \Big[\frac{1}{2} q_0 + \frac{3}{8}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^2 + \frac{5}{24}\left(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{7}{24}\left(3f_2 - f_1^2 - 24f_1f_2 - 60f_2^2\right)\left(\frac{r}{a}\right)^6 + \frac{9}{40}\left(-f_2 + 16f_1f_2 + 80f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{11}{15}\left(f_1f_2 - 15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{26}{7} f_2^2 \left(\frac{r}{a}\right)^{12} - \frac{15}{28} f_2^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \left\{ -\frac{15}{96}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 + \frac{14}{20} f_2 \left(\frac{r}{a}\right)^6 - \frac{15}{80} f_2 \left(\frac{r}{a}\right)^8 \right.$$

$$\left. + 2p_2 + 12 r_2 \left(\frac{r}{a}\right)^2 \right\}$$

$$\widehat{r\theta} = E\left(\frac{a}{R}\right)^2 \sin 2\theta \left\{ -\frac{5}{96}\left(f_1 + 6f_2\right)\left(\frac{r}{a}\right)^4 + \frac{7}{40} f_2 \left(\frac{r}{a}\right)^6 - \frac{3}{80} f_2 \left(\frac{r}{a}\right)^8 + 2p_2 + 6 r_2 \left(\frac{r}{a}\right)^2 \right\}$$

$$\sigma_r - \sigma_\theta = E\left(\frac{a}{R}\right)^2\Big[\frac{1}{2}(1-\nu)q_0 + \frac{(1-3\nu)}{8}\left(f_1+2f_2-2f_1^2-8f_1f_2-8f_2^2\right)\left(\frac{r}{a}\right)^2 + \frac{(1-5\nu)}{24}\left(-f_1-6f_2+4f_1^2+32f_1f_2+48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+\frac{(1-7\nu)}{24}\left(3f_2-f_1^2-24f_1f_2-60f_2^2\right)\left(\frac{r}{a}\right)^6+\frac{(1-9\nu)}{40}\left(-f_2+16f_1f_2+80f_2^2\right)\left(\frac{r}{a}\right)^8+\frac{(1-11\nu)}{15}\left(f_1f_2-15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+\frac{2(1-13\nu)}{7}f_2^2\left(\frac{r}{a}\right)^{12}-\frac{(1-15\nu)}{28}f_2^2\left(\frac{r}{a}\right)^{14}+\cos 2\theta\Big\{-\frac{(1-15\nu)}{96}\left(f_1+6f_2\right)\left(\frac{r}{a}\right)^4+\frac{(1-14\nu)}{20}f_2\left(\frac{r}{a}\right)^6-\frac{(1-15\nu)}{80}f_2\left(\frac{r}{a}\right)^8$$

$$-2(1+\nu)p_2-12\nu r_2\left(\frac{r}{a}\right)^2\Big\}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2=\frac{1}{2}\left(\frac{r}{R}\right)^2\Big[(\cos 2\theta-1)\Big\{2f_1\left[1-\left(\frac{r}{a}\right)^2\right]+4f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\Big\}$$

$$+4f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]^2+16f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^4+16f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^6\Big]$$

$$=\left(\frac{a}{R}\right)^2\Big[-f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2-2f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left(\frac{r}{a}\right)^2+2f_1^2\left[1-\left(\frac{r}{a}\right)^2\right]^2\left(\frac{r}{a}\right)^2+8f_1f_2\left[1-\left(\frac{r}{a}\right)^2\right]^4\left(\frac{r}{a}\right)^2+8f_2^2\left[1-\left(\frac{r}{a}\right)^2\right]^6\left(\frac{r}{a}\right)^2$$

$$+\cos 2\theta\Big\{f_1\left[1-\left(\frac{r}{a}\right)^2\right]\left(\frac{r}{a}\right)^2+2f_2\left[1-\left(\frac{r}{a}\right)^2\right]^3\left(\frac{r}{a}\right)^2\Big\}\Big]$$

$$=\left(\frac{a}{R}\right)^2\Big[-\left(f_1+2f_2-2f_1^2-8f_1f_2-8f_2^2\right)\left(\frac{r}{a}\right)^2-\left(-f_1-6f_2+4f_1^2+32f_1f_2+48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$-2\left(3f_2-f_1^2-24f_1f_2-60f_2^2\right)\left(\frac{r}{a}\right)^6-9\left(-f_2+16f_1f_2+80f_2^2\right)\left(\frac{r}{a}\right)^8-8\left(f_1f_2-15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$-48f_2^2\left(\frac{r}{a}\right)^{12}+8f_2^2\left(\frac{r}{a}\right)^{14}+\cos 2\theta\Big\{\left(f_1+2f_2\right)\left(\frac{r}{a}\right)^2-\left(f_1+6f_2\right)\left(\frac{r}{a}\right)^4+6f_2\left(\frac{r}{a}\right)^6-2f_2\left(\frac{r}{a}\right)^8\Big\}\Big]$$

$$\frac{\partial u}{\partial r} = \left(\frac{a}{R}\right)^2 \Big[\frac{1}{2}(1-\nu) q_0 + \frac{3(3-\nu)}{8}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^2 + \frac{5(5-\nu)}{24}\left(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2\right)\left(\frac{r}{a}\right)^4$$

$$+ \frac{7(7-\nu)}{24}\left(3f_2 - f_2^2 - 24f_1f_2 - 60f_2^2\right)\left(\frac{r}{a}\right)^6 + \frac{9(9-\nu)}{40}\left(-f_2 + 16f_1f_2 + 80f_2^2\right)\left(\frac{r}{a}\right)^8 + \frac{11(11-\nu)}{15}\left(f_1f_2 - 15f_2^2\right)\left(\frac{r}{a}\right)^{10}$$

$$+ \frac{26(13-\nu)}{7} f_2^2 \left(\frac{r}{a}\right)^{12} - \frac{15(15-\nu)}{28} f_2^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ -(f_1 + 2f_2)\left(\frac{r}{a}\right)^2 + \frac{(95+15\nu)}{96}(f_1 + 6f_2)\left(\frac{r}{a}\right)^4 - \frac{(119+14\nu)}{20} f_2 \left(\frac{r}{a}\right)^6$$

$$+ \frac{(159+15\nu)}{80} f_2 \left(\frac{r}{a}\right)^8$$

$$- 2(1+\nu) p_2 - 12\nu h_2 \left(\frac{r}{a}\right)^2 \Big\} \Big]$$

$$\frac{u}{R} = \left(\frac{a}{R}\right)^3 \Big[\frac{1}{2}(1-\nu) q_0 \left(\frac{r}{a}\right) + \frac{(3-\nu)}{8}\left(f_1 + 2f_2 - 2f_1^2 - 8f_1f_2 - 8f_2^2\right)\left(\frac{r}{a}\right)^3 + \frac{(5-\nu)}{24}\left(-f_1 - 6f_2 + 4f_1^2 + 32f_1f_2 + 48f_2^2\right)\left(\frac{r}{a}\right)^5$$

$$+ \frac{(7-\nu)}{24}\left(3f_2 - f_2^2 - 24f_1f_2 - 60f_2^2\right)\left(\frac{r}{a}\right)^7 + \frac{(9-\nu)}{40}\left(-f_2 + 16f_1f_2 + 80f_2^2\right)\left(\frac{r}{a}\right)^9 + \frac{(11-\nu)}{15}\left(f_1f_2 - 15f_2^2\right)\left(\frac{r}{a}\right)^{11}$$

$$+ \frac{2(13-\nu)}{7} f_2^2 \left(\frac{r}{a}\right)^{12} - \frac{(15-\nu)}{28} f_2^2 \left(\frac{r}{a}\right)^{14} + \cos 2\theta \Big\{ -\frac{1}{3}(f_1 + 2f_2)\left(\frac{r}{a}\right)^3 + \frac{(19+3\nu)}{96}(f_1 + 6f_2)\left(\frac{r}{a}\right)^5 - \frac{(17+2\nu)}{20} f_2 \left(\frac{r}{a}\right)^7$$

$$+ \frac{(53+5\nu)}{240} f_2 \left(\frac{r}{a}\right)^9 - 2(1+\nu) p_2 \left(\frac{r}{a}\right) - 4\nu h_2 \left(\frac{r}{a}\right)^3 \Big\} \Big]$$

Non-uniform part of $\widehat{\theta\theta} - \nu\widehat{rr}$ is

$$E\left(\frac{a}{R}\right)^2 \cos 2\theta \left\{ -\frac{(15-\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^4 + \frac{(14-\nu)}{20} f_2 \left(\frac{r}{a}\right)^6 - \frac{(45-3\nu)}{240} f_2 \left(\frac{r}{a}\right)^8 + 2(1+\nu)p_2 + 12 h_2 \left(\frac{r}{a}\right)^2 \right\}$$

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \left(\frac{a}{R}\right)^2 \cos 2\theta \left[\frac{1}{3}(f_1+2f_2)\left(\frac{r}{a}\right)^2 - \frac{(34+2\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^4 + \frac{(31+\nu)}{20} f_2 \left(\frac{r}{a}\right)^6 - \frac{(98+2\nu)}{240} f_2 \left(\frac{r}{a}\right)^9 \right.$$

$$\left. + 4(1+\nu)p_2 + 4(3+\nu)h_2 \left(\frac{r}{a}\right)^2 \right\} \Big]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \sin 2\theta \left\{ \frac{1}{6}(f_1+2f_2)\left(\frac{r}{a}\right)^3 - \frac{(17+\nu)}{96}(f_1+6f_2)\left(\frac{r}{a}\right)^5 + \frac{(31+\nu)}{40} f_2 \left(\frac{r}{a}\right)^7 - \frac{(49+\nu)}{240} f_2 \left(\frac{r}{a}\right)^9 \right.$$

$$\left. + 2(1+\nu)p_2\left(\frac{r}{a}\right) + 2(3+\nu)h_2\left(\frac{r}{a}\right)^3 \right\} \Big]$$

$$\widehat{rr}_a = \cos 2\theta \left\{ -\frac{1}{96} f_1 - \frac{1}{40} f_2 - 2p_2 \right\}$$

$$\widehat{\theta\theta}_a = \cos 2\theta \left\{ -\frac{5}{32} f_1 - \frac{17}{40} f_2 + 2p_2 + 12 h_2 \right\}$$

$$\widehat{r\theta}_a = \sin 2\theta \left\{ -\frac{5}{96} f_1 - \frac{7}{40} f_2 + 2p_2 + 6 h_2 \right\}$$

$$\left(\frac{u}{R}\right)_a = \cos 2\theta \left\{ -\frac{(13-3\nu)}{96} f_1 - \frac{26-26\nu}{140} f_2 - 2(1+\nu)p_2 - 12\nu h_2 \right\}$$

$$\left(\frac{v}{R}\right)_a = \sin 2\theta \left\{ -\frac{(11+\nu)}{96} f_1 - \frac{38+10\nu}{140} f_2 + 2(1+\nu)p_2 + 2(3+\nu)h_2 \right\}$$

$$\frac{1}{2} - 6q_2 - 4s_2 = \eta\left\{-\frac{1}{96}f_1 - \frac{1}{40}f_2 - 2p_2\right\}$$

$$6q_2 - \frac{1}{2} = \eta\left\{-\frac{5}{32}f_1 - \frac{17}{40}f_2 + 2p_2 + 12r_2\right\}$$

$$\frac{1}{2} + 6q_2 + 2s_2 = \eta\left\{+\frac{5}{96}f_1 + \frac{7}{40}f_2 - 2p_2 - 6r_2\right\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2 = \eta\left\{-\frac{(13-3\nu)}{96}f_1 - \frac{26(1-\nu)}{240}f_2 - 2(1+\nu)p_2 - 4\nu r_2\right\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \eta\left\{-\frac{(1+\nu)}{96}f_1 - \frac{38+10\nu}{240}f_2 + 2(1+\nu)p_2 + 2(3+\nu)r_2\right\}$$

$\dfrac{12}{2496}$ $\dfrac{41}{6240}$

$$q_2 + 0.66666667s_2 - 0.08333333 = \eta\{+0.3333333p_2 + 0 + 0.00173611f_1 + 0.00416667f_2\}$$

$$q_2 \quad 0 \quad -0.08333333 = \eta\{+0.3333333p_2 + 2r_2 - 0.02604167f_1 - 0.07083333f_2\}$$

$$q_2 + 0.3333333s_2 + 0.08333333 = \eta\{-0.3333333p_2 - r_2 + 0.00868056f_1 + 0.02916667f_2\}$$

$$q_2 + 1.53846154s_2 + 0.2500000 = \eta\{-p_2 - 0.46153846r_2 - 0.04847756f_1 - 0.07916667f_2\}$$

$$q_2 + 0 - 0.2500000 = \eta\{+p_2 + 2.53846154r_2 - 0.00520833f_1 - 0.06570513f_2\}$$

$$0.66666667s_2 + 0 = \eta\{0 \quad -2r_2 + 0.02777778f_1 + 0.07500000f_2\}$$

$$0.333333s_2 + 0.1666667 = \eta\{-0.666667p_2 - 3r_2 + 0.03472222f_1 + 0.10000000f_2\}$$

$$1.20512821s_2 + 0.166667 = \eta\{-0.666667p_2 + 0.53846154r_2 - 0.05715812f_1 - 0.058333333f_2\}$$

$$1.53846154s_2 + 0.50000 = \eta\{-2p_2 - 3r_2 - 0.04326923f_1 + 0.03653846f_2\}$$

$$s_2 + 0 = \eta\{0 - 3r_2 + 0.04166667 f_1 + 0.11250000 f_2\}$$
$$s_2 + 0.50000 = \eta\{-2p_2 - 9r_2 + 0.10416667 f_1 + 0.3000000 f_2\}$$
$$s_2 + 0.13829787 = \eta\{-0.55319149 p_2 + 0.44680851 r_2 - 0.04742908 f_1 - 0.04840425 f_2\}$$
$$s_2 + 0.3250000 = \eta\{-1.3 p_2 - 1.95 r_2 - 0.02812500 f_1 + 0.02375000 f_2\}$$

$$0.5000000 = \eta\{-2p_2 - 6r_2 + 0.06250000 f_1 + 0.18750000 f_2\}$$
$$0.36170213 = \eta\{-1.44680851 p_2 - 9.44680851 r_2 + 0.15159575 f_1 + 0.34840425 f_2\}$$
$$0.18670213 = \eta\{-0.74680851 p_2 - 2.39680851 r_2 + 0.01930408 f_1 + 0.07215425 f_2\}$$

$$0.2500000 = \eta\{-p_2 - 3r_2 + 0.03125000 f_1 + 0.09375000 f_2\}$$
$$0.2500000 = \eta\{-p_2 - 6.52941176 r_2 + 0.10477942 f_1 + 0.24080882 f_2\}$$
$$0.250000 = \eta\{-p_2 - 3.20940171 r_2 + 0.02584877 f_1 + 0.09661680 f_2\}$$

$$3.52941176 r_2 = 0.07352942 f_1 + 0.14705882 f_2$$
$$3.32001005 r_2 = 0.07893065 f_1 + 0.14419202 f_2$$

$$3r_2 = 0.06250000 f_1 + 0.12500000 f_2$$
$$3r_2 = 0.07132266 f_1 + 0.13029360 f_2$$

$$0 = 0.00882266 f_1 + 0.00529360 f_2$$

$$f_2 = -\frac{5}{3} f_1$$

failure !!!...

1)

$$\left(\frac{w}{R}\right)_0 = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta\right\}$$

$$\left(\frac{w}{R}\right) = \frac{1}{2}\left(\frac{a}{R}\right)^2\left\{1-\left(\frac{r}{a}\right)^2\sin^2\theta - f_4\left[1-\left(\frac{r}{a}\right)^2\right]^6\right\}$$

$$\frac{1}{r}\frac{\partial w}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^5\right\}$$

$$\frac{1}{r}\frac{\partial w_0}{\partial r} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta\right\}$$

$$\frac{\partial^2 w}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^5 - 120 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left(\frac{r}{a}\right)^2\right\}$$

$$= \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-\left(\frac{r}{a}\right)^2 - 10\left(\frac{r}{a}\right)^2\right]\right\}$$

$$= \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta + 12 f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$\frac{\partial^2 w_0}{\partial r^2} = \frac{1}{2}\frac{1}{R}\left\{-2\sin^2\theta\right\}$$

$$\frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} = \frac{1}{r^2}\frac{\partial^2 w_0}{\partial\theta^2} = \frac{1}{R}\left(-\cos 2\theta\right)$$

$$-\left\{\frac{1}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}-\frac{1}{r}\frac{\partial w_0}{\partial r}\frac{\partial^2 w_0}{\partial r^2}\right\}$$

6)

$$=\frac{1}{R^2}(\sin^2\theta)^2-\frac{1}{R^2}\left\{\sin^2\theta-6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^5\right\}\left\{\sin^2\theta-6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$=\frac{1}{R^2}\left\{\underline{\underline{(1-\cos 2\theta)\cdot 6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-6\left(\frac{r}{a}\right)^2\right]}}-36f_4^2\left[1-\left(\frac{r}{a}\right)^2\right]^9\left[1-11\left(\frac{r}{a}\right)^2\right]\right\}$$

$$-\left\{\frac{1}{r^2}\frac{\partial^2 w}{\partial r^2}\frac{\partial^2 w}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial^2 w_0}{\partial r^2}\frac{\partial^2 w_0}{\partial\theta^2}\right\}$$

$$=\frac{1}{R^2}\cos 2\theta\left\{\underline{\underline{6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left[1-11\left(\frac{r}{a}\right)^2\right]}}\right\}$$

The non-uniform terms in particular integral must be derived from

$$\cos 2\theta\left[-6f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4 5\left(\frac{r}{a}\right)^2\right]=-\left[30f_4\left[1-\left(\frac{r}{a}\right)^2\right]^4\left(\frac{r}{a}\right)^2\right]\cos 2\theta$$

$$=-30f_4\cos 2\theta\left\{\left(\frac{r}{a}\right)^2-4\left(\frac{r}{a}\right)^4+6\left(\frac{r}{a}\right)^6-4\left(\frac{r}{a}\right)^8+\left(\frac{r}{a}\right)^{10}\right\}$$

$$\frac{\Phi}{R^2}=-30f_4\cos 2\theta\left\{\frac{1}{32\cdot 12}\left(\frac{r}{a}\right)^6-\frac{4}{60\cdot 32}\left(\frac{r}{a}\right)^8+\frac{6}{96\cdot 60}\left(\frac{r}{a}\right)^{10}-\frac{4}{140\cdot 96}\left(\frac{r}{a}\right)^{12}+\frac{1}{192\cdot 140}\left(\frac{r}{a}\right)^{14}\right\}E\left(\frac{a}{R}\right)^7$$

$$\frac{1}{r}\frac{\partial \Phi}{\partial r} = -30 f_4 \cos 2\theta \left\{ \frac{1}{64}\left(\frac{r}{a}\right)^4 - \frac{1}{60}\left(\frac{r}{a}\right)^6 + \frac{1}{96}\left(\frac{r}{a}\right)^8 - \frac{1}{280}\left(\frac{r}{a}\right)^{10} + \frac{1}{1920}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2 \qquad c)$$

$$\frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = +30 f_4 \cos 2\theta \left\{ \frac{1}{96}\left(\frac{r}{a}\right)^4 - \frac{1}{120}\left(\frac{r}{a}\right)^6 + \frac{1}{240}\left(\frac{r}{a}\right)^8 - \frac{1}{840}\left(\frac{r}{a}\right)^{10} + \frac{1}{48\times 140}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\hat{\sigma}_{rr} = -30 f_4 \cos 2\theta \left\{ \frac{1}{192}\left(\frac{r}{a}\right)^4 - \frac{1}{120}\left(\frac{r}{a}\right)^6 + \frac{1}{160}\left(\frac{r}{a}\right)^8 - \frac{1}{420}\left(\frac{r}{a}\right)^{10} + \frac{1}{192\cdot 14}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\hat{\sigma}_{\theta\theta} = -30 f_4 \cos 2\theta \left\{ \frac{5}{64}\left(\frac{r}{a}\right)^4 - \frac{7}{60}\left(\frac{r}{a}\right)^6 + \frac{9}{96}\left(\frac{r}{a}\right)^8 - \frac{11}{280}\left(\frac{r}{a}\right)^{10} + \frac{13}{1920}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\hat{\sigma}_{r\theta} = -30 f_4 \sin 2\theta \left\{ \frac{5}{32\cdot 6}\left(\frac{r}{a}\right)^4 - \frac{7^{(X)}}{60\cdot 8}\left(\frac{r}{a}\right)^6 + \frac{9}{96\times 5}\left(\frac{r}{a}\right)^8 - \frac{11}{140\cdot 12}\left(\frac{r}{a}\right)^{10} + \frac{13}{192\times 70}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\hat{\sigma}_{rr} - \nu\hat{\sigma}_{\theta\theta} = -30 f_4 \cos 2\theta \left\{ \frac{(1-15\nu)}{192}\left(\frac{r}{a}\right)^4 - \frac{(1-14\nu)}{120}\left(\frac{r}{a}\right)^6 + \frac{(1-15\nu)}{160}\left(\frac{r}{a}\right)^8 - \frac{(2-33\nu)}{840}\left(\frac{r}{a}\right)^{10} + \frac{(5-91\nu)}{13440}\left(\frac{r}{a}\right)^{12} \right\} E\left(\frac{a}{R}\right)^2$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2 = \frac{1}{2}\left(\frac{r}{R}\right)^2 \left\{ -(1-\cos 2\theta)\, 6 f_4 \left[1-\left(\frac{r}{a}\right)^2\right]^5 + 36 f_4^2 \left[1-\left(\frac{r}{a}\right)^2\right]^{10} \right\}$$

$$= \frac{1}{2}\left(\frac{a}{R}\right)^2 \left\{ \cos 2\theta \cdot 6 f_4 \left[1-\left(\frac{r}{a}\right)^2\right]^5 \left(\frac{r}{a}\right)^2 + \cdots\cdots \right\}$$

the non-uniform part of $\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial r}\right)^2$ d)

$$3f_4 \cos 2\theta \left\{ \left(\frac{r}{a}\right)^2 - 5\left(\frac{r}{a}\right)^4 + 10\left(\frac{r}{a}\right)^6 - 10\left(\frac{r}{a}\right)^8 + 5\left(\frac{r}{a}\right)^{10} - \left(\frac{r}{a}\right)^{12} \right\} \left(\frac{a}{R}\right)^2$$

$$\frac{\partial u}{\partial r} = -30 f_4 \cos 2\theta \left\{ \frac{1}{10}\left(\frac{r}{a}\right)^2 - \frac{(95+15\nu)}{192}\left(\frac{r}{a}\right)^4 + \frac{(119+14\nu)}{120}\left(\frac{r}{a}\right)^6 - \frac{(159+15\nu)}{160}\left(\frac{r}{a}\right)^8 \right.$$

$$\left. + \frac{(418+33\nu)}{840}\left(\frac{r}{a}\right)^{10} - \frac{(1339+91\nu)}{13440}\left(\frac{r}{a}\right)^{12} \right\} \left(\frac{a}{R}\right)^2$$

$$\frac{u}{R} = -30 f_4 \cos 2\theta \left\{ \frac{1}{30}\left(\frac{r}{a}\right)^3 - \frac{(19+3\nu)}{192}\left(\frac{r}{a}\right)^5 + \frac{(17+2\nu)}{120}\left(\frac{r}{a}\right)^7 - \frac{(53+5\nu)}{480}\left(\frac{r}{a}\right)^9 \right.$$

$$\left. + \frac{(38+3\nu)}{840}\left(\frac{r}{a}\right)^{11} - \frac{(103+7\nu)}{13440}\left(\frac{r}{a}\right)^{13} \right\} \left(\frac{a}{R}\right)^3$$

$$\frac{u}{r} = -30 f_4 \cos 2\theta \left\{ \frac{1}{30}\left(\frac{r}{a}\right)^2 - \frac{(19+3\nu)}{192}\left(\frac{r}{a}\right)^4 + \frac{(17+2\nu)}{120}\left(\frac{r}{a}\right)^6 - \frac{(53+5\nu)}{480}\left(\frac{r}{a}\right)^8 \right.$$

$$\left. + \frac{(38+3\nu)}{840}\left(\frac{r}{a}\right)^{10} - \frac{(103+7\nu)}{13440}\left(\frac{r}{a}\right)^{12} \right\} \left(\frac{a}{R}\right)^2$$

$$\frac{1}{E}\left(\sigma_\theta - \sigma_r\right) = -30 f_4 \cos 2\theta \left\{ \frac{(15-\nu)}{192}\left(\frac{r}{a}\right)^4 - \frac{(14-\nu)}{120}\left(\frac{r}{a}\right)^6 + \frac{(15-\nu)}{160}\left(\frac{r}{a}\right)^8 - \frac{(33-2\nu)}{840}\left(\frac{r}{a}\right)^{10} \right.$$

$$\left. + \frac{(91-5\nu)}{13440}\left(\frac{r}{a}\right)^{12} \right\} \left(\frac{a}{R}\right)^2$$

$$\frac{\partial v}{r\partial \theta} = -30 f_4 \cos 2\theta \left\{ -\frac{1}{30}\left(\frac{r}{a}\right)^2 + \frac{(34+2\nu)}{192}\left(\frac{r}{a}\right)^4 - \frac{(31+\nu)}{120}\left(\frac{r}{a}\right)^6 + \frac{(49+\nu)}{240}\left(\frac{r}{a}\right)^8 \right.$$

$$\left. - \frac{(71+\nu)}{840}\left(\frac{r}{a}\right)^{10} + \frac{(97+\nu)}{6720}\left(\frac{r}{a}\right)^{12} \right\} \left(\frac{a}{R}\right)^2$$

e)

$$\frac{\tau}{h} = -30 f_4 \sin 2\theta \left\{ -\frac{1}{60}\left(\frac{r}{a}\right)^3 + \frac{(17+\nu)}{192}\left(\frac{r}{a}\right)^5 - \frac{(31+\nu)}{240}\left(\frac{r}{a}\right)^7 + \frac{(49+\nu)}{480}\left(\frac{r}{a}\right)^9 - \frac{(71+\nu)}{1680}\left(\frac{r}{a}\right)^{11} + \frac{(97+\nu)}{13440}\left(\frac{r}{a}\right)^{13} \right\} \left(\frac{a}{h}\right)^2$$

$$\frac{1}{2} - 6q_2 - 4s_2 = \eta\{-2p_2 - 0.03348214 f_4\}$$

$$6q_2 - \frac{1}{2} = \eta\{+2p_2 + 12r_2 - 0.27455357 f_4\}$$

$$-\frac{1}{2} - 6q_2 - 2s_2 = \eta\{+2p_2 + 6r_2 - 0.68080357 f_4\}$$

$$\frac{1}{2}(1+\nu) + 2(1+\nu)q_2 + 4s_2 = \eta\{-2(1+\nu)p_2 - 4\nu r_2 - 0.044797642 f_4\}$$

$$2(1+\nu)q_2 - \frac{1}{2}(1+\nu) = \eta\{+2(1+\nu)p_2 + 2(3+\nu)r_2 - 0.31584822 f_4\}$$

$$q_2 + 0.6666667 s_2 - 0.0833333 = \eta\{+0.333333 p_2 + 0 + 0.005580356 f_4\}$$

$$q_2 - 0.0833333 = \eta\{+0.333333 p_2 + 2r_2 - 0.045758928 f_4\}$$

$$q_2 + 0.333333 s_2 + 0.083333 = \eta\{-0.33333 p_2 - r_2 + 0.113467262 f_4\}$$

$$q_2 + 1.53846154 s_2 + 0.250000 = \eta\{-p_2 - 0.461538462 r_2 - 0.01722870 f_4\}$$

$$q_2 - 0.250000 = \eta\{+p_2 + 2.53846154 r_2 - 0.12148008 f_4\}$$

f)

$$0.66666667\, S_2 = \eta \{ 0 - 2r_2 + 0.05133929 f_4 \}$$

$$0.3333333\, S_2 + 0.1666667 = \eta \{ -0.6666667\, p_2 - 3r_2 + 0.15922619 f_4 \}$$

$$1.20512821\, S_2 + 0.166667 = \eta \{ -0.166667\, p_2 + 0.53846154\, r_2 - 0.13069597 f_4 \}$$

$$1.53846154\, S_2 + 0.50000 = \eta \{ -2p_2 - 3r_2 + 0.10425137 f_4 \}$$

$$S_2 + 0 = \eta \{ 0 - 3r_2 + 0.077001935 f_4 \}$$

$$S_2 + 0.5000 = \eta \{ -2p_2 - 9r_2 + 0.47767857 f_4 \}$$

$$S_2 + 0.13829787 = \eta \{ -0.55319149\, p_2 + 0.44680851\, r_2 - 0.10844985 f_4 \}$$

$$S_2 + 0.325000 = \eta \{ -1.30\, p_2 - 1.95\, r_2 + 0.06776339 f_4 \}$$

$$0.500000000 = \eta \{ -2p_2 - 6r_2 + 0.40066963 f_4 \}$$

$$0.36170213 = \eta \{ -1.44680851\, p_2 - 9.44680851\, r_2 + 0.58612842 f_4 \}$$

$$0.18670213 = \eta \{ -0.74680851\, p_2 - 2.39680851\, r_2 + 0.17621324 f_4 \}$$

$$0.2500000 = \eta \{ -p_2 - 3r_2 + 0.10033482 f_4 \}$$

$$0.250000 = \eta \{ -p_2 - 6.52941175\, r_2 + 0.40511818 f_4 \}$$

$$0.25000 = \eta \{ -p_2 - 3.20940171\, r_2 + 0.23595505 f_4 \}$$

$3.52941175\, r_2 = 0.20478336\, f_4$ $0.17406585\, 6\, f_4$ 8)

$3.32001005\, r_2 = 0.16916313\, f_4$

JOURNAL OF THE AERONAUTICAL SCIENCES

20th and Northampton Streets, Easton, Pa.

Postmaster: Please send undeliverable copies and notices to

5111 RCA Building
Rockefeller Center
New York, N. Y.

RETURN POSTAGE GUARANTEED

Printed in United States of America

Entered as second class mail matter at the Easton, Pa., Post Office

Shell VI

Hsue-Shen [illegible]
Aeronautics Department
Calif. Inst. of Technology
Pasadena, Calif.

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2 - \frac{f}{4}\left(1+\cos\frac{\pi x}{a}\right)\left(1+\cos\frac{\lambda\pi y}{a}\right)\right]$$

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$R\frac{\partial^2 w}{\partial x^2} = \left[-1+\frac{f}{8}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi\lambda y}{a}\right)\right] \qquad R\frac{\partial^2 w_0}{\partial x^2} = -1$$

$$R\frac{\partial^2 w}{\partial y^2} = \left[+\frac{f}{8}\lambda^2\pi^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi\lambda y}{a}\right]$$

$$R\frac{\partial^2 w}{\partial x\partial y} = \left[-\frac{f}{8}\lambda\pi^2\sin\frac{\pi x}{a}\sin\frac{\pi\lambda y}{a}\right]$$

$$\nabla^4 F = \pi^2\frac{E}{R^2}\left(\frac{f\lambda^2}{8}\right)\left[\frac{f\pi^2}{8}\frac{1}{4}\left(1-\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi\lambda y}{a}\right)\right.$$
$$+\left(1+\cos\frac{\pi x}{a}\right)\left(\cos\frac{\pi\lambda y}{a}\right) - \frac{f\pi^2}{8}\frac{1}{4}\left(1+2\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)$$
$$\left.\left(1+2\cos\frac{\pi\lambda y}{a}+\cos\frac{2\pi\lambda y}{a}\right)\right]$$

$$= \frac{E}{R^2}\left(\frac{f\lambda^2}{8}\right)\pi^2\left[\cos\frac{\pi\lambda y}{a}+\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}\right.$$
$$+\frac{f\pi^2}{32}\left(\cancel{1}-2\cos\frac{2\pi x}{a}-2\cos\frac{2\pi\lambda y}{a}+\cancel{\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}}\right.$$
$$-\cancel{1}-2\cos\frac{\pi x}{a}-\cancel{\cos\frac{2\pi x}{a}}-2\cos\frac{\pi\lambda y}{a}-4\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}-2\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}$$
$$\left.\left.-\cancel{\cos\frac{2\pi\lambda y}{a}}-2\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}-\cancel{\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}}\right)\right]$$

$$= \pi^2\frac{E}{R^2}\left(\frac{f\lambda^2}{8}\right)\left[-\frac{f\pi^2}{16}\cos\frac{\pi x}{a}+\left(1-\frac{f\pi^2}{16}\right)\cos\frac{\pi\lambda y}{a}+\left(1-\frac{f\pi^2}{8}\right)\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}\right.$$
$$\left.-\frac{f\pi^2}{16}\left[\cos\frac{2\pi x}{a}+\cos\frac{2\pi\lambda y}{a}+\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}+\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right]\right]$$

$$F = E\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[-\frac{f\pi^2}{16}\frac{\cos\frac{\pi x}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{\lambda^4}\left(1-\frac{f\pi^2}{16}\right)\frac{\cos\frac{\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{(1+\lambda^2)^2}\left(1-\frac{f\pi^2}{8}\right)\frac{\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2}$$

$$-\frac{f\pi^2}{16}\left\{\frac{1}{16}\frac{\cos\frac{2\pi x}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{16\lambda^4}\frac{\cos\frac{2\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{(1+4\lambda^2)^2}\frac{\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2} + \frac{1}{(4+\lambda^2)^2}\frac{\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}}{\left(\frac{\pi}{a}\right)^2}\right\}$$

$$+\frac{1}{\left(\frac{\pi}{a}\right)^2}\left\{a_1\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\lambda\pi y}{a}\right)\sinh\frac{\lambda\pi y}{a}\right\}\cos\frac{\pi x}{a}$$

$$+\frac{1}{\left(\frac{2\pi}{a}\right)^2}\left\{a_2\cosh\left(\frac{2\pi\lambda y}{a}\right) + b_2\left(\frac{2\lambda\pi y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}\Bigg] + \frac{\sigma}{2}x^2$$

$$\frac{\sigma_y}{E} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[-\left\{a_1\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a} - \left\{a_2\cosh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}$$

$$+\frac{f\pi^2}{16}\cos\frac{\pi x}{a} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{16}\left\{\frac{1}{4}\cos\frac{2\pi x}{a} + \frac{1}{(1+4\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left.+\frac{4}{(4+\lambda^2)^2}\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}\right\}\Bigg] + \frac{\sigma}{E}$$

$$a_1\cosh\pi + \pi b_1\sinh\pi = \frac{f\pi^2}{16} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right) + \frac{1}{(1+\lambda^2)^2}\frac{f\pi^2}{16}$$

$$(a_1+b_1)\sinh\pi + \pi b_1\cosh\pi = 0$$

$$a_2\cosh 2\pi + 9\pi b_2\sinh 2\pi = \frac{f\pi^2}{16}\left[\frac{1}{4} + \frac{4}{(4+\lambda^2)^2}\right]$$

$$(a_2+b_2)\sinh 2\pi + 2\pi b_2\cosh 2\pi = 0$$

452

$$\frac{\sigma_y}{E}=\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[\left\{\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi\lambda y}{a}+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\cos\frac{2\pi\lambda y}{a}-a_1\cosh\frac{\pi\lambda y}{a}-b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a}\right.$$

$$\left.+\left\{\frac{f\pi^2}{16}\frac{1}{4}+\frac{f\pi^2}{16}\frac{4}{(4+\lambda^2)^2}\cos\frac{\pi\lambda y}{a}-a_2\cosh\frac{2\pi\lambda y}{a}-b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}\right]-\frac{\sigma}{E}$$

$$\frac{1}{aE^2}\int_0^a\sigma_y^2\,dx=\frac{1}{2}\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\left[\left\{\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi y}{b}+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\cos\frac{2\pi y}{b}\right\}-\left\{a_1\cosh\frac{\pi y}{b}+b_1\left(\frac{\pi y}{b}\right)\sinh\frac{\pi y}{b}\right\}\right]^2$$

$$+\frac{1}{2}\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\left[\left\{\frac{f\pi^2}{16}\frac{1}{4}+\frac{f\pi^2}{16}\frac{4}{(4+\lambda^2)^2}\cos\frac{\pi y}{b}\right\}-\left\{a_2\cosh\frac{2\pi y}{b}+b_2\left(\frac{2\pi y}{b}\right)\sinh\frac{2\pi y}{b}\right\}\right]^2+\left(\frac{\sigma}{E}\right)^2$$

$$\frac{1}{abE^2}\int_0^a\int_0^b\sigma_y^2\,dx\,dy=\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\left[\frac{1}{2}\left(\frac{f\pi^2}{16}\right)^2+\frac{1}{4}\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2+\frac{1}{4}\frac{1}{(1+4\lambda^2)^4}\left(\frac{f\pi^2}{16}\right)^2+\frac{1}{2}\left(\frac{f\pi^2}{64}\right)^2+\frac{1}{4}\frac{1}{(4+\lambda^2)^4}\left(\frac{f\pi^2}{4}\right)^2\right]$$

$$-\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\left[\frac{f\pi^2}{16}\frac{1}{\pi}\int_0^\pi(a_1\cosh u+b_1u\sinh u)\,du+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\frac{1}{\pi}\int_0^\pi\cos u\,(a_1\cosh u+b_1u\sinh u)\,du\right.$$

$$+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\frac{1}{\pi}\int_0^\pi\cos 2u\,(a_1\cosh u+b_1u\sinh u)\,du+\frac{f\pi^2}{64}\frac{1}{\pi}\int_0^\pi(a_2\cosh 2u+b_2(2u)\sinh 2u)\,du$$

$$\left.+\frac{f\pi^2}{4(4+\lambda^2)^2}\frac{1}{\pi}\int_0^\pi\cos u\,(a_2\cosh 2u+b_2\,2u\sinh 2u)\,du\right]$$

$$+\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\left[\frac{1}{2}\frac{1}{\pi}\int_0^\pi(a_1\cosh u+b_1u\sinh u)^2\,du+\frac{1}{2\pi}\int_0^\pi(a_2\cosh 2u+b_1\,2u\sinh 2u)^2\,du\right]$$

We have the following integrals:

$$\frac{1}{\pi}\int_0^\pi (\alpha \cosh u + \beta u \sinh u)\,du = (\alpha-\beta)\frac{\sinh\pi}{\pi} + \beta\cosh\pi$$

$$\frac{1}{\pi}\int_0^\pi \cos u\,(\alpha\cosh u + \beta u \sinh u)\,du = -\alpha\,\frac{\sinh\pi}{\pi} - \beta\cosh\pi$$

$$\frac{1}{\pi}\int_0^\pi \cos 2u\,(\alpha\cosh u + \beta u\sinh u)\,du = \frac{\alpha}{5}\,\frac{\sinh\pi}{\pi} + \beta\left(\frac{1}{5}\cosh\pi + \frac{3}{25}\,\frac{\sinh\pi}{\pi}\right)$$

$$\frac{1}{\pi}\int_0^\pi (\gamma\cosh 2u + \delta\, 2u \sinh 2u)\,du = (\gamma-\delta)\,\frac{\sinh 2\pi}{2\pi} + \delta\cosh 2\pi$$

$$\frac{1}{\pi}\int_0^\pi \cos u\,(\gamma\cosh 2u + \delta\, 2u\sinh 2u)\,du = \frac{4}{5}\gamma\,\frac{\sinh 2\pi}{2\pi} + \delta\left(\frac{4}{5}\cosh 2\pi + \frac{12}{25}\,\frac{\sinh 2\pi}{2\pi}\right)$$

$$\frac{1}{\pi}\int_0^\pi \left\{\alpha\cosh u + \beta u\sinh u\right\}^2 du$$

$$= \frac{\alpha^2}{2}\left(\frac{\sinh 2\pi}{2\pi} + 1\right) + \frac{\alpha\beta}{2}\left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi}\right) + \frac{\beta^2}{2}\left\{\left(\frac{2\pi^2+1}{2}\right)\frac{\sinh 2\pi}{2\pi} - \frac{1}{2}\cosh 2\pi - \frac{\pi^2}{3}\right\}$$

$$\frac{1}{\pi}\int_0^\pi \left\{\gamma\cosh 2u + \delta\,2u\sinh 2u\right\}^2 du$$

$$= \frac{\gamma^2}{2}\left(\frac{\sinh 4\pi}{4\pi} + 1\right) + \frac{\gamma\delta}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{\delta^2}{2}\left\{\left(\frac{8\pi^2+1}{2}\right)\frac{\sinh 4\pi}{4\pi} - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right\}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin u \sinh u \, du = \frac{1}{2}\,\frac{\sinh\pi}{\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} \sin u \cdot u \cosh u \, du = \frac{1}{2}\left(\cosh\pi - \frac{\sinh\pi}{\pi}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} \sin 2u \cdot \sinh u \, du = -\frac{2}{5}\left(\frac{\sinh\pi}{\pi}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} \sin u \sinh 2u \, du = \frac{2}{5}\left(\frac{\sinh 2\pi}{2\pi}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} 2u \cdot \sin u \cosh 2u \, du = \frac{2}{5}\cosh 2\pi - \frac{16}{25}\,\frac{\sinh 2\pi}{2\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} u \cdot \sin 2u \cosh u \, du = -\frac{2}{5}\cosh\pi + \frac{4}{25\pi}\,\frac{\sinh\pi}{\pi}$$

$$\frac{1}{\pi}\int_0^{\pi} (\alpha \sinh u + \beta u \cosh u)^2 du$$

$$= \frac{\alpha^2}{2}\left(\frac{\sinh 2\pi}{2\pi} - 1\right) + \frac{\alpha\beta}{2}\left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi}\right) + \frac{\beta^2}{2}\left(\frac{2\pi^2+1}{2}\,\frac{\sinh 2\pi}{2\pi} - \frac{1}{2}\cosh 2\pi + \frac{\pi^2}{3}\right)$$

$$\frac{1}{\pi}\int_0^{\pi} (\gamma \sinh 2u + \delta u \cosh 2u)^2 du$$

$$= \frac{\gamma^2}{2}\left(\frac{\sinh 4\pi}{4\pi} - 1\right) + \frac{\gamma\delta}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{\delta^2}{2}\left(\frac{8\pi^2+1}{2}\,\frac{\sinh 4\pi}{4\pi} - \frac{1}{2}\cosh 4\pi + \frac{4\pi^2}{3}\right)$$

Therefore

$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{h}\right)^2\left(\frac{f\lambda^2}{8}\right)^2\left[\frac{1}{2}\left(\frac{f\pi^2}{16}\right)^2 + \frac{1}{4(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2 + \frac{1}{4(1+4\lambda^2)^4}\left(\frac{f\pi^2}{16}\right)^2 + \frac{1}{2}\left(\frac{f\pi^2}{64}\right)^2 + \frac{1}{4(4+\lambda^2)^4}\left(\frac{f\pi^2}{4}\right)^2\right.$$

$$-\left\{\frac{f\pi^2}{16}\left[(a_1-b_1)\frac{\sinh\pi}{\pi} + b_1\cosh\pi\right] + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\left[-a_1\frac{\sinh\pi}{\pi} - b_1\cosh\pi\right]\right.$$

$$+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{16}\left[a_1\frac{1}{5}\frac{\sinh\pi}{\pi} + b_1\left(\frac{1}{5}\cosh\pi + \frac{3}{25}\frac{\sinh\pi}{\pi}\right)\right] + \frac{f\pi^2}{64}\left[(a_2-b_2)\frac{\sinh 2\pi}{2\pi} + b_2\cosh 2\pi\right]$$

$$\left.+\frac{(f\pi^2)}{4(4+\lambda^2)^2}\left[-\frac{4}{5}a_2\frac{\sinh 2\pi}{2\pi} + b_2\left(-\frac{4}{5}\cosh 2\pi + \frac{12}{25}\frac{\sinh 2\pi}{2\pi}\right)\right]\right\}$$

$$+\frac{1}{4}a_1^2\left(\frac{\sinh 2\pi}{2\pi}+1\right) + \frac{a_1b_1}{4}\left(\cosh 2\pi - \frac{\sinh 2\pi}{2\pi}\right) + \frac{b_1^2}{4}\left(\frac{2\pi^2+1}{2}\frac{\sinh 2\pi}{2\pi} - \frac{1}{2}\cosh 2\pi - \frac{\pi^2}{3}\right)$$

$$\left.+\frac{1}{4}a_2^2\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{a_2b_2}{4}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{4}\left(\frac{8\pi^2+1}{2}\frac{\sinh 4\pi}{4\pi} - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right]$$

456

$$\frac{\sigma_x}{E} = \lambda^2\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{(a_1+2b_1)\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a} + \left\{(a_2+2b_2)\cosh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}$$

$$+ \frac{1}{\lambda^4}\left(\frac{f\pi^2}{16}-1\right)\cos\frac{\pi\lambda y}{a} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{16}\left\{\frac{1}{4\lambda^4}\cos\frac{2\pi\lambda y}{a} + \frac{4}{(1+4\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left. + \frac{1}{(4+\lambda^2)^2}\cos\frac{2\pi x}{a}\cos\frac{\pi\lambda y}{a}\right\}\Bigg]$$

$$= \lambda^2\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{\frac{1}{\lambda^4}\left(\frac{f\pi^2}{16}-1\right)\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{64\lambda^4}\cos\frac{2\pi\lambda y}{a}\right\}$$

$$+\left\{\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi\lambda y}{a} + \frac{f\pi^2}{4(1+4\lambda^2)^2}\cos\frac{2\pi\lambda y}{a} + (a_1+2b_1)\cosh\frac{\pi\lambda y}{a} + b_1\left(\frac{\pi\lambda y}{a}\right)\sinh\frac{\pi\lambda y}{a}\right\}\cos\frac{\pi x}{a}$$

$$+\left\{\frac{f\pi^2}{16(4+\lambda^2)^2}\cos\frac{\pi\lambda y}{a} + (a_2+2b_2)\cosh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right\}\cos\frac{2\pi x}{a}\Bigg]$$

$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_x^2\,dx\,dy = \lambda^4\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\frac{1}{2}\frac{1}{\lambda^8}\left(\frac{f\pi^2}{16}-1\right)^2+\frac{1}{2}\frac{1}{\lambda^8}\left(\frac{f\pi^2}{64}\right)^2+\frac{1}{4}\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2+\frac{1}{4(1+4\lambda^2)^4}\left(\frac{f\pi^2}{4}\right)^2$$

$$+\frac{1}{4(4+\lambda^2)^4}\left(\frac{f\pi^2}{16}\right)^2$$

$$+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\left\{-(a_1+2b_1)\frac{\sinh\pi}{\pi}-b_1\cosh\pi\right\}+\frac{f\pi^2}{4(1+4\lambda^2)^2}\left\{\frac{a_1+2b_1}{5}\frac{\sinh\pi}{\pi}+b_1\left(\frac{1}{5}\cosh\pi+\frac{3}{25}\frac{\sinh\pi}{\pi}\right)\right\}$$

$$+\frac{f\pi^2}{16(4+\lambda^2)^2}\left\{-\frac{4}{5}(a_2+2b_2)\frac{\sinh 2\pi}{2\pi}+b_2\left(-\frac{4}{5}\cosh 2\pi+\frac{12}{25}\frac{\sinh 2\pi}{2\pi}\right)\right\}$$

$$+\frac{1}{4}(a_1+2b_1)^2\left(\frac{\sinh 2\pi}{2\pi}+1\right)+\frac{(a_1+2b_1)b_1}{4}\left(\cosh 2\pi-\frac{\sinh 2\pi}{2\pi}\right)+\frac{b_1^2}{4}\left(\frac{2\pi^2+1}{2}\frac{\sinh 2\pi}{2\pi}-\frac{1}{2}\cosh 2\pi-\frac{\pi^2}{3}\right)$$

$$+\frac{1}{4}(a_2+2b_2)^2\left(\frac{\sinh 4\pi}{4\pi}+1\right)+\frac{(a_2+2b_2)b_2}{4}\left(\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}\right)+\frac{b_2^2}{4}\left(\frac{8\pi^2+1}{2}\frac{\sinh 4\pi}{4\pi}-\frac{1}{2}\cosh 4\pi-\frac{4\pi^2}{3}\right)\Bigg]$$

$$\frac{\tau_{xy}}{E} = \lambda\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{(a_1+b_1)\sinh\frac{\pi\lambda y}{a} + b_1\left(\frac{\lambda\pi y}{a}\right)\cosh\frac{\lambda\pi y}{a}\right\}\sin\frac{\pi x}{a} + \left\{(a_2+b_2)\sinh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\cosh\frac{2\pi\lambda y}{a}\right\}\sin\frac{2\pi x}{a}$$

$$+ \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\sin\frac{\pi x}{a}\sin\frac{\pi\lambda y}{a} + \frac{f\pi^2}{8}\frac{1}{(1+4\lambda^2)^2}\sin\frac{\pi x}{a}\sin\frac{2\pi\lambda y}{a} + \frac{f\pi^2}{8}\frac{1}{(4+\lambda^2)^2}\sin\frac{2\pi x}{a}\sin\frac{\pi\lambda y}{a}\Bigg]$$

$$= \lambda\left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\Bigg[\left\{\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\sin\frac{\pi\lambda y}{a} + \frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{8}\sin\frac{2\pi\lambda y}{a} + (a_1+b_1)\sinh\frac{\pi\lambda y}{a} + b_1\left(\frac{\lambda\pi y}{a}\right)\cosh\frac{\lambda\pi y}{a}\right\}\sin\frac{\pi x}{a}$$

$$+ \left\{\frac{1}{(4+\lambda^2)^2}\left(\frac{f\pi^2}{8}\right)\sin\frac{\pi\lambda y}{a} + (a_2+b_2)\sinh\frac{2\pi\lambda y}{a} + b_2\left(\frac{2\pi\lambda y}{a}\right)\cosh\frac{2\pi\lambda y}{a}\right\}\sin\frac{2\pi x}{a}\Bigg]$$

459

$$\frac{1}{abE^2}\int_0^a\int_0^b \tau_{xy}^2\,dy\,dx = \lambda^2\left(\frac{a}{R}\right)^4\left(\frac{f\lambda^2}{8}\right)^2\Bigg[\frac{1}{4(1+\lambda^2)^4}\left(\frac{f\pi^2}{8}-1\right)^2+\frac{1}{4(1+4\lambda^2)^4}\left(\frac{f\pi^2}{8}\right)^2+\frac{1}{4(4+\lambda^2)^2}\left(\frac{f\pi^2}{8}\right)^2$$

$$+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)\left\{\frac{1}{2}(a_1+b_1)\frac{\sinh\pi}{\pi}+\frac{b_1}{2}\left(\cosh\pi-\frac{\sinh\pi}{\pi}\right)\right\}+\frac{1}{(1+4\lambda^2)^2}\frac{f\pi^2}{8}\left\{-\frac{2}{5}(a_1+b_1)\frac{\sinh\pi}{\pi}+b_1\left(-\frac{2}{5}\cosh\pi+\frac{4}{25\pi}\frac{\sinh\pi}{\pi}\right)\right\}$$

$$+\frac{1}{(4+\lambda^2)^2}\left(\frac{f\pi^2}{8}\right)\left\{\frac{2}{5}(a_2+b_2)\frac{\sinh 2\pi}{2\pi}+b_2\left(\frac{2}{5}\cosh 2\pi-\frac{16}{25}\frac{\sinh 2\pi}{2\pi}\right)\right\}$$

$$+\frac{(a_1+b_1)^2}{4}\left(\frac{\sinh 2\pi}{2\pi}-1\right)+\frac{(a_1+b_1)b_1}{4}\left(\cosh 2\pi-\frac{\sinh 2\pi}{2\pi}\right)+\frac{b_1^2}{4}\left(\frac{2\pi^2+1}{2}\frac{\sinh 2\pi}{2\pi}-\frac{1}{2}\cosh 2\pi+\frac{\pi^2}{3}\right)$$

$$+\frac{(a_2+b_2)^2}{4}\left(\frac{\sinh 4\pi}{4\pi}-1\right)+\frac{(a_2+b_2)b_2}{4}\left(\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}\right)+\frac{b_2^2}{4}\left(\frac{8\pi^2+1}{2}\frac{\sinh 4\pi}{4\pi}-\frac{1}{2}\cosh 4\pi+\frac{4\pi^2}{3}\right)\Bigg]$$

46

$$F_1 = (f\pi^2)^2 \Big\{ \frac{1}{512}\frac{1}{\lambda^4} + \frac{1}{8192}\frac{1}{\lambda^4} + \frac{1}{256}\frac{\lambda^4}{(1+\lambda^2)^4} + \frac{\lambda^4}{64(1+4\lambda^2)^4} + \frac{\lambda^4}{1024(4+\lambda^2)^4}$$

$$+ \frac{1}{512} + \frac{1}{8192} + \frac{1}{256}\frac{1}{(1+\lambda^2)^4} + \frac{1}{1024(1+4\lambda^2)^4} + \frac{1}{64(4+\lambda^2)^2} + \frac{2\lambda^2}{256(1+\lambda^2)^4} + \frac{2\lambda^2}{256(1+4\lambda^2)}$$

$$+ \frac{2\lambda^2}{256(4+\lambda^2)^2} \Big\}$$

$$- (f\pi^2) \Big\{ \frac{1}{16}\frac{1}{\lambda^4} + \frac{\lambda^4}{16(1+\lambda^2)^4} + \frac{1}{16(1+\lambda^2)^4} + \frac{2\lambda^2}{16(1+\lambda^2)^2} \Big\} + \Big\{ \frac{1}{2\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{1}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4} \Big\}$$

$$= (f\pi^2)^2 \Big\{ \frac{17}{8192}\left(1+\frac{1}{\lambda^4}\right) + \frac{1}{256}\frac{1}{(1+\lambda^2)^2} + \frac{1}{1024(1+4\lambda^2)^2} + \frac{1}{1024(4+\lambda^2)^2} \Big\}$$

$$- (f\pi^2) \Big\{ \frac{1}{16\lambda^4} + \frac{1}{16(1+\lambda^2)^2} \Big\} + \Big\{ \frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2} \Big\}$$

a, $\frac{\sinh \pi}{\pi}$

$$-\frac{\lambda^4}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)+\frac{\lambda^4}{20(1+4\lambda^2)^2}(f\pi^2)-\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)$$

$$-\frac{1}{80(1+4\lambda^2)^2}(f\pi^2)+\frac{\lambda^2}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)-\frac{\lambda^2(f\pi^2)}{20(1+4\lambda^2)^2}=g_1(\lambda)$$

---

b, $\frac{\sinh \pi}{\pi}$

$$-\frac{2\lambda^4}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)+\frac{13}{25}\frac{\lambda^4}{4(1+4\lambda^2)^2}(f\pi^2)+\frac{f\pi^2}{16}-\frac{3}{25}\frac{1}{16(1+4\lambda^2)^2}(f\pi^2)$$

$$-\frac{3}{25}\frac{\lambda^2 f\pi^2}{4(1+4\lambda^2)^2}=g_2(\lambda)$$

---

b, $\cosh \pi$

$$-\frac{\lambda^4}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)+\frac{\lambda^4 f\pi^2}{20(1+4\lambda^2)^2}-\frac{f\pi^2}{16}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8}-1\right)-\frac{f\pi^2}{80(1+4\lambda^2)^2}$$

$$+\frac{\lambda^2}{(1+\lambda^2)^2}-\frac{\lambda^2}{10(1+4\lambda^2)^2}(f\pi^2)=g_1(\lambda)$$

---

a_2 $\frac{\sinh 2\pi}{2\pi}$

$$-\frac{\lambda^4(f\pi^2)}{20(4+\lambda^2)^2}-\frac{f\pi^2}{64}+\frac{f\pi^2}{5(4+\lambda^2)^2}+\frac{\lambda^2(f\pi^2)}{10(4+\lambda^2)^2}=g_3(\lambda)$$

$b_2 \dfrac{\sinh 2\pi}{2\pi}$

$$-\frac{7\lambda^4}{25\times 4(4+\lambda^2)^2}(f\pi^2)+\frac{f\pi^2}{64}-\frac{12}{100}\frac{(f\pi^2)}{(4+\lambda^2)^2}-\frac{3\times 2\lambda^2}{25\times 4(4+\lambda^2)^2}(f\pi^2)=g_4(\lambda)$$

$b_2 \cosh 2\pi$

$$-\frac{\lambda^4(f\pi^2)}{20(4+\lambda^2)^2}-\frac{f\pi^2}{64}+\frac{(f\pi^2)}{5(4+\lambda^2)^2}+\frac{4\lambda^2(f\pi^2)}{40(4+\lambda^2)^2}=g_3(\lambda)$$

$$\cosh\pi\, a_1+\pi\sinh\pi\, b_1=\left\{\frac{1}{16}+\frac{1}{8(1+\lambda^2)^2}+\frac{1}{16(1+4\lambda^2)^2}\right\}(f\pi^2)-\frac{1}{(1+\lambda^2)^2}$$

$$\sinh\pi\, a_1+(\sinh\pi+\pi\cosh\pi)\, b_1=0$$

$H_1(\lambda)$ $H_2(\lambda)$

$$b_1=-\frac{\frac{\sinh\pi}{\pi}}{\frac{\sinh 2\pi}{2\pi}+1}\left[\left\{\frac{1}{16}+\frac{1}{8(1+\lambda^2)^2}+\frac{1}{16(1+4\lambda^2)^2}\right\}f\pi^2-\frac{1}{(1+\lambda^2)^2}\right]$$

$$a_1=+\frac{\frac{\sinh\pi}{\pi}+\cosh\pi}{\frac{\sinh 2\pi}{2\pi}+1}\left[\left\{\frac{1}{16}+\frac{1}{8(1+\lambda^2)^2}+\frac{1}{16(1+4\lambda^2)^2}\right\}f\pi^2-\frac{1}{(1+\lambda^2)^2}\right]$$

$H_3(\lambda)$

$$a_2=+\frac{\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi}{\frac{\sinh 4\pi}{4\pi}+1}\left[\frac{1}{64}+\frac{1}{4(4+\lambda^2)^2}\right]f\pi^2$$

$$b_2=-\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi}+1}\left[\frac{1}{64}+\frac{1}{4(4+\lambda^2)^2}\right]f\pi^2$$

Thus

$$F_2 = -\left[H_1 f\pi^2 - H_2\right]\left[-\frac{\sinh\pi}{\pi} g_1\left(\frac{\sinh\pi}{\pi} + \cosh\pi\right)\right.$$

$$\left. + \left(\frac{\sinh\pi}{\pi}\right)^2 g_2 + \frac{\sinh\pi}{\pi}\cdot\cosh\pi\; g_1\right]\frac{1}{\frac{\sinh 2\pi}{2\pi}+1}$$

$$-\left[H_3 (f\pi^2)^2\right]\left[-g_3\frac{\sinh 2\pi}{2\pi}\left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi\right) + \left(\frac{\sinh 2\pi}{2\pi}\right)^2 g_4 + g_3\frac{\sinh 2\pi}{2\pi}\cosh 2\pi\right]$$

$$\times \frac{1}{\frac{\sinh 4\pi}{4\pi}+1}$$

$$= -\frac{\left(\frac{\sinh\pi}{\pi}\right)^2}{\frac{\sinh 2\pi}{2\pi}+1}\left[H_1(f\pi^2) - H_2\right]\left[g_2 - g_1\right]$$

$$-\frac{\left(\frac{\sinh 2\pi}{2\pi}\right)^2}{\frac{\sinh 4\pi}{4\pi}+1} H_3 (f\pi^2)^2\left[g_4 - g_3\right]$$

$$g_2 - g_1 = -\frac{\lambda^2}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8} - 1\right) + \frac{2\lambda^2}{25(1+4\lambda^2)^2}(f\pi^2) + \frac{1}{8}(f\pi^2) - \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8} - 1\right)$$

$$+\frac{1}{200(1+4\lambda^2)^2}(f\pi^2) - \frac{\lambda^2}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{8} - 1\right) + \frac{\lambda^2}{25(1+4\lambda^2)^2}(f\pi^2)$$

$$= \left(\frac{f\pi^2}{8} - 1\right)\left(\frac{\lambda^2}{(1+\lambda^2)^2} - 1\right) + \frac{1}{8}(f\pi^2) + (f\pi^2)\left(\frac{1}{200} - \frac{\lambda^2}{25(1+4\lambda^2)^2}\right)$$

$$= (f\pi^2)\left\{\frac{1}{200} + \frac{\lambda^2}{8(1+\lambda^2)^2} - \frac{\lambda^2}{25(1+4\lambda^2)^2}\right\} + \left\{1 - \frac{\lambda^2}{(1+\lambda^2)^2}\right\}$$

$$g_4 - g_3 = -\frac{\lambda^4}{50(4+\lambda^2)^2} + \frac{1}{32} - \frac{8}{25(4+\lambda^2)^2} - \frac{8\lambda^2}{50(4+\lambda^2)^2}$$

$$= \frac{1}{32} - \frac{1}{50} = \frac{9}{800}$$

$$H_1(\lambda) = \frac{1}{8}\left\{\frac{1}{2} + \frac{1}{(1+\lambda^2)^2} + \frac{1}{2(1+4\lambda^2)^2}\right\}$$

$$H_2(\lambda) = \frac{1}{(1+\lambda^2)^2}$$

$$H_3(\lambda) = \frac{1}{4}\left\{\frac{1}{16} + \frac{1}{(4+\lambda^2)^2}\right\}$$

$$H_4(\lambda) = \frac{1}{200} + \frac{\lambda^2}{8(1+\lambda^2)^2} - \frac{\lambda^2}{25(1+4\lambda^2)^2}$$

$$H_5(\lambda) = 1 - \frac{\lambda^2}{(1+\lambda^2)^2}$$

$$F_2 = -\frac{\left(\frac{\sinh\pi}{\pi}\right)^2}{\frac{\sinh 2\pi}{2\pi}+1}\left[H_1(\sqrt{\pi^2}) - H_2\right]\left[H_4(\text{[illegible]}) + H_{[illegible]}\right]$$

$$-\frac{9}{800}\,\frac{\left(\frac{\sinh 2\pi}{2\pi}\right)^2}{\frac{\sinh 4\pi}{4\pi}+1}\,H_3(\sqrt{\pi^2})^2$$

$$F_3 = \left[H_1(1+\pi^2) - H_2\right]^2 \times \frac{1}{\left(\frac{\sinh 2\pi}{2\pi}+1\right)^2}$$

$$\Bigg[\frac{1}{4}\left(\frac{\sinh\pi}{\pi}+\cosh\pi\right)^2\left\{(1+\lambda^2)^2\frac{\sinh 2\pi}{2\pi}+(1-\lambda^2)^2\right\}$$

$$-\frac{1}{4}\frac{\sinh\pi}{\pi}\left(\frac{\sinh\pi}{\pi}+\cosh\pi\right)\left\{(3\lambda^4+2\lambda^2-1)\frac{\sinh 2\pi}{2\pi}+(1+\lambda^2)^2\cosh 2\pi-4(1-\lambda^2)\lambda^2\right\}$$

$$+\frac{1}{4}\left(\frac{\sinh\pi}{\pi}\right)^2\Bigg\{\left[\pi^2(1+\lambda^2)^2+\frac{1}{2}(5\lambda^4+2\lambda^2+1)\right]\frac{\sinh 2\pi}{2\pi}+\frac{1}{2}(3\lambda^4+2\lambda^2-1)\cosh 2\pi$$

$$-\left[\frac{\pi^2}{3}(1-\lambda^2)^2+2(1-2\lambda^4)\right]\Bigg\}\Bigg]$$

$$+\ H_3^2(1+\pi^2)^2 \times \frac{1}{\left(\frac{\sinh 4\pi}{4\pi}+1\right)^2}$$

$$\Bigg[\frac{1}{4}\left(\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi\right)^2\left\{(1+\lambda^2)^2\frac{\sinh 4\pi}{4\pi}+(1-\lambda^2)^2\right\}$$

$$-\frac{1}{4}\frac{\sinh 2\pi}{2\pi}\left(\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi\right)\left\{(3\lambda^4+2\lambda^2-1)\frac{\sinh 4\pi}{4\pi}+(1+\lambda^2)^2\cosh 4\pi-4(1-\lambda^2)\lambda^2\right\}$$

$$+\frac{1}{4}\left(\frac{\sinh 2\pi}{2\pi}\right)^2\Bigg\{\left[4\pi^2(1+\lambda^2)^2+\frac{1}{2}(5\lambda^4+2\lambda^2+1)\right]\frac{\sinh 4\pi}{4\pi}+\frac{1}{2}(3\lambda^4+2\lambda^2-1)\cosh 2\pi$$

$$-\left[\frac{4\pi^2}{3}(1-\lambda^2)^2+2(1-2\lambda^4)\right]\Bigg\}\Bigg]$$

$$\frac{1}{12}\frac{1}{ab}t^2\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2}-\frac{\partial^2 w_0}{\partial x^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{8}\right)^2\frac{3}{4}$$

$$\frac{1}{12}\frac{1}{ab}t^2\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2\lambda^2}{8}\right)^2\frac{3}{4}$$

$$\frac{1}{12}\frac{1}{ab}t^2\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2\lambda}{8}\right)^2\frac{1}{4}$$ $\times 2$!

the bending energy $/(E\,ab)t(\frac{1}{2})$

$$= \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f}{8}\right)^2\pi^4\left[\frac{3}{4}(1+\lambda^4)+\frac{1}{2}\lambda^2\right]$$

$$= \left(\frac{t}{R}\right)^2\left(\frac{f}{8}\right)^2\pi^4\left[\frac{1}{16}(1+\lambda^4)+\frac{1}{24}\lambda^2\right]$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \frac{1}{2}\left[\frac{f\lambda\pi}{8}\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\lambda\pi y}{a}\right]^2\left(\frac{a}{R}\right)^2$$

$$= \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[\frac{f\pi^2}{16}\frac{1}{4}\left(3+2\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\left(1-\cos\frac{2\pi\lambda y}{a}\right)\right]$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[-\frac{3}{4}\frac{f\pi^2}{16}+\cdots\right]-\frac{\sigma}{E}$$

$$\left(\frac{v}{b}\right)_{y=b} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda^2}{8}\right)\left[-\frac{3}{4}\frac{f\pi^2}{16}+\cdots\right]-\frac{\sigma}{E}$$

$$2\times\frac{\Delta\wp}{Eabt} = -\frac{\sigma}{E}\left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)^2\frac{3}{32}(f\pi^2) - 2\left(\frac{\sigma}{E}\right)^2$$

$$= -\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)^2\lambda^2\frac{3}{4}\pi^2\frac{\sigma}{E} - 2\left(\frac{\sigma}{E}\right)^2$$

---

for limiting case of $\lambda << 1$

$$\frac{3}{2}\pi^2\frac{\sigma}{E} = \frac{1}{\lambda^2}\left(\frac{a}{R}\right)^2\left\{\frac{17}{2048}\pi^4 f^2 - \frac{3}{16}\pi^2 f + 1\right\} + \frac{1}{8}\pi^4\frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\frac{1}{\lambda^2}$$

1024

$$\lambda^2 K = \rho^2\left\{\frac{17}{3072}\pi^2 f^2 - \frac{1}{8}f + \frac{2}{3\pi^2}\right\} + \frac{1}{12}\frac{\pi^2}{\rho^2}$$

$$= \frac{\pi^2}{\rho^2}\left\{\frac{17}{768}\left(\frac{\delta}{t}\right)^2 + \frac{1}{12}\right\} - \frac{1}{4}\left(\frac{\delta}{t}\right) + \frac{2}{3}\frac{\rho^2}{\pi^2}$$

$$= 2\left\{\frac{17}{1152}\left(\frac{\delta}{t}\right)^2 + \frac{1}{18}\right\} - \frac{1}{4}\left(\frac{\delta}{t}\right)$$

$$\left(\frac{1}{64} - \frac{17}{1152}\right)\left(\frac{\delta}{t}\right)^2 = \frac{1}{18} \qquad \left(\frac{\delta}{t}\right)^2 = \frac{1}{18}\times 1152 \qquad \left(\frac{\delta}{t}\right) = 8$$

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2-\frac{f}{4}\left(1+\cos\frac{\pi x}{a}\right)\left(1+\cos\frac{\pi y}{b}\right)\right]$$

469

×

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$\sigma_x = E\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right\}$$

$$\sigma_y = E\left\{\frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$\tau_{xy} = E\left\{\frac{1}{2}\frac{\partial v}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right\}$$

By using the equilibrium equation $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, we have

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -\left\{\frac{1}{2}\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial y} + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y} + \frac{\partial w}{\partial y}\frac{\partial^2 w}{\partial y^2}\right\}$$

$$= -\left\{\frac{1}{2}\frac{\partial w}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + 2\frac{\partial^2 w}{\partial y^2}\right) + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x \partial y}\right\}$$

We have

$$\frac{\partial w}{\partial x} = \left(\frac{a}{R}\right)\left[-\left(\frac{x}{a}\right) + \frac{f}{8}\pi \sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)\left[\frac{f}{8}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{R}\left[-1 + \frac{f}{8}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{\partial^2 w}{\partial y^2} = \frac{1}{R}\left[\frac{f}{8}\pi^2\left(\frac{a}{b}\right)^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{R}\left[-\frac{f}{8}\pi^2\left(\frac{a}{b}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$R\left(\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right) = -\left(\frac{a}{R}\right)\Big[\frac{f}{g}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\Big\{\frac{f}{g}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)+\frac{f}{g}\pi^2\left(\frac{a}{b}\right)^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}-1\Big\}$$

$$-\Big\{\frac{f}{g}\pi\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)-\left(\frac{x}{a}\right)\Big\}\frac{f}{g}\pi^2\left(\frac{q}{b}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\Big]$$

$$= -\left(\frac{q}{R}\right)\Big[\left(\frac{f}{g}\right)^2\pi^3\left(\frac{q}{b}\right)\frac{1}{4}\left(x+2\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)$$

$$+\left(\frac{f}{g}\right)^2\pi^3\left(\frac{q}{b}\right)^3\frac{1}{4}\left(3+4\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\sin\frac{2\pi y}{b}-\frac{f}{g}\pi\left(\frac{q}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}$$

$$-\left(\frac{f}{g}\right)^2\pi^3\left(\frac{q}{b}\right)\frac{1}{4}\left(1-\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)+\frac{f}{g}\pi^2\left(\frac{q}{b}\right)\left(\frac{x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\Big]$$

$$= -\left(\frac{q}{R}\right)\left(\frac{f}{g}\right)\pi\left(\frac{q}{b}\right)\Big[\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\left(4\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+4\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+2\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+2\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right)$$

$$+\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\left(\frac{q}{b}\right)^2\left(3\sin\frac{2\pi y}{b}+4\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right)$$

$$-\Big\{1+\cos\frac{\pi x}{a}-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\Big\}\sin\frac{\pi y}{b}\Big]$$

$$= -\left(\frac{q}{R}\right)\left(\frac{f}{g}\right)\pi\left(\frac{q}{b}\right)\Big[\frac{1}{4}\left(\frac{f}{g}\right)\pi^2\Big\{3\left(\frac{q}{b}\right)^2\sin\frac{2\pi y}{b}+4\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+4\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+\left(2+4\left(\frac{q}{b}\right)^2\right)\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}$$

$$+\left(2+\left(\frac{q}{b}\right)^2\right)\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$-\Big\{\sin\frac{\pi y}{b}+\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\Big\}\Big]$$

Consider $\quad R\left(\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2}\right) = \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}$

Let

$$v = X \sin\frac{\pi y}{b}$$

$$R\left[\frac{d^2 X}{dx^2} - 2\left(\frac{\pi}{b}\right)^2 X\right] = \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a}$$

Let

$$X = A \cos\frac{\pi x}{a} + B\left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a}$$

$$\frac{dX}{dx} = -\left(\frac{\pi}{a}\right) A \sin\frac{\pi x}{a} + \left(\frac{\pi}{a}\right) B \sin\frac{\pi x}{a} + \left(\frac{\pi}{a}\right) B \left(\frac{\pi x}{a}\right) \cos\frac{\pi x}{a}$$

$$\frac{d^2 X}{dx^2} = -\left(\frac{\pi}{a}\right)^2 (A - 2B) \cos\frac{\pi x}{a} - \left(\frac{\pi}{a}\right)^2 B \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a}$$

$$R\left[\left\{-\left(\frac{\pi}{a}\right)^2 (A-2B) - 2\left(\frac{\pi}{b}\right)^2 A\right\} \cos\frac{\pi x}{a} - \left\{\left(\frac{\pi}{a}\right)^2 B + 2\left(\frac{\pi}{b}\right)^2 B\right\} \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a}\right] = \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a}$$

$$\therefore \quad B = -\frac{1}{\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2} \frac{1}{R}$$

$$\left[\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2\right] A = 2\left(\frac{\pi}{a}\right)^2 B, \qquad A = \frac{2\left(\frac{\pi}{a}\right)^2}{\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2} B = -\frac{2\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2 + 2\left(\frac{\pi}{b}\right)^2\right]^2} \frac{1}{R}$$

The particular integral :

$$\frac{v}{R} = +\left(\frac{a}{R}\right)\left(\frac{f}{g}\right)\pi\left(\frac{a}{b}\right)\frac{1}{R^2}\Bigg[\frac{f}{4}\left(\frac{f}{g}\right)\pi^2\Bigg\{\frac{3\left(\frac{a}{b}\right)^2}{2\left(\frac{2\pi}{b}\right)^2}\sin\frac{2\pi y}{b} + \frac{4}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{4}{\left(\frac{2\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}$$

$$+\frac{2+4\left(\frac{a}{b}\right)^2}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{2\pi}{b}\right)^2}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{2+\left(\frac{a}{b}\right)^2}{\left(\frac{2\pi}{a}\right)^2+2\left(\frac{2\pi}{b}\right)^2}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\Bigg\}$$

$$-\Bigg\{\frac{1}{2\left(\frac{\pi}{b}\right)^2}\sin\frac{\pi y}{b} + \frac{1\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{2\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$-\frac{1}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\Bigg\}\Bigg]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3\left(\frac{f}{g}\right)\pi\lambda\Bigg[\frac{f}{4}\left(\frac{f}{g}\right)\Bigg\{\frac{3}{8}\sin\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{4}{(4+2\lambda^2)}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}$$

$$+\frac{2+4\lambda^2}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{2+\lambda^2}{4+8\lambda^2}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\Bigg\}$$

$$-\frac{1}{\pi^3}\Bigg\{\frac{1}{2\lambda^2}\sin\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\Bigg\}$$

$$+\frac{1}{\pi}a_0\left(\frac{\pi y}{b}\right) + \frac{1}{\pi}a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}\Bigg]$$

Add the complementary function.

$$\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial^2 v}{\partial y^2} = 0 , \qquad A_0\left(\frac{\pi y}{b}\right) + A_1 \cosh\frac{\sqrt{2}\,\pi x}{b} \sin\frac{\pi y}{b}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2 \left[\frac{f\pi^2}{32}\left\{\frac{3}{4}\cos\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{4}{(4+2\lambda^2)}\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b}\right.\right.$$

$$\left. + \frac{4(1+2\lambda^2)}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b} + \frac{2+\lambda^2}{2(2+2\lambda^2)}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\right\}$$

$$-\left\{\frac{1}{2\lambda^2}\cos\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\right\}$$

$$\left. + a_0 + a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\cos\frac{\pi y}{b}\right]$$

$$\left(\frac{\partial v}{\partial y}\right)_{y=\pm b} = \left(\frac{\sigma}{E}\right)_{y=\pm b} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2\left[\frac{f\pi^2}{32}\left\{\frac{3}{4} - \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a} - \frac{4}{(4+2\lambda^2)}\cos\frac{2\pi x}{a} + \frac{4(1+2\lambda^2)}{1+8\lambda^2}\cos\frac{\pi x}{a}\right.\right.$$

$$\left. + \frac{2+\lambda^2}{2(2+2\lambda^2)}\cos\frac{2\pi x}{a}\right\} + \left\{\frac{1}{2\lambda^2} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\right\}$$

$$\left. + a_0 - a_1\cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\right]$$

Average stress $= \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2\left[\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2} + a_0\right] = -\frac{\sigma}{E}$

The decrease in potential $\qquad \frac{\Delta \wp}{abEt} = \left[\left(\frac{a}{R}\right)^3\left(\frac{f}{8}\right) \lambda^2\right]^2 a_0\left[\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2} + a_0\right]$

$$\boxed{\frac{\Delta \wp}{abEt} = +\frac{\sigma}{E}\left\{\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right) \lambda^2\left(\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2}\right)\right\}}$$

Decrease in Potential

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3\left(\frac{f}{8}\right)\pi\lambda\left[\frac{f}{32}\left\{\frac{3}{8}\sin\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{2}{(2+\lambda^2)}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}\right.\right.$$

$$\left. + \frac{2(1+2\lambda^2)}{(1+8\lambda^2)}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{2+\lambda^2}{4(1+2\lambda^2)}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}$$

$$-\frac{1}{\pi^2}\left\{\frac{1}{2\lambda^2}\sin\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\right\}$$

$$\left. - \left(\frac{3}{128} f + \frac{1}{\pi^2 \cdot 2\lambda^2}\right)\left(\frac{\pi y}{b}\right) + \frac{1}{\pi} a_1 \cosh\frac{\sqrt{2}\,\lambda\pi x}{a}\sin\frac{\pi y}{b}\right] - \frac{\sigma}{E}\left(\frac{y}{R}\right)$$

$$\frac{\partial v}{\partial x} = \left(\frac{a}{h}\right)^2 \left(\frac{f}{g}\right) \lambda \Big[\frac{f\pi^2}{32} \Big\{ -\frac{4}{(1+2\lambda^2)} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} - \frac{4}{(2+\lambda^2)} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} - \frac{2(1+2\lambda^2)}{(1+8\lambda^2)} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b}$$

$$- \frac{2+\lambda^2}{2(1+2\lambda^2)} \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{b} \Big\} + \Big\{ \frac{\text{[illegible]}(1+\lambda^2)}{(1+2\lambda^2)^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)} \sin\left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b}$$

$$+ \frac{1}{(1+2\lambda^2)} \left(\frac{\pi x}{a}\right) \cos\left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b} \Big\}$$

$$+ \sqrt{2}\,\lambda\, a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} \sin\frac{\pi y}{b} \Big]$$

$\left(\frac{\partial v}{\partial x}\right) = 0$ when $x = \pm a$,

$$\frac{1}{(1+2\lambda^2)}\pi = \sqrt{2}\,\lambda\, a_1 \sinh(\sqrt{2}\lambda\pi)$$

$$\boxed{a_1 = \frac{\pi}{(1+2\lambda^2)\sqrt{2}\,\lambda \sinh(\sqrt{2}\lambda\pi)}}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2 \left(\frac{f}{8}\right) \lambda^2 \Bigg[\frac{f\pi^2}{32} \Big\{ \frac{3}{4} \cos\frac{2\pi y}{b} + \frac{4}{(1+2\lambda^2)} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{2}{(9+\lambda^2)} \cos\frac{2\pi x}{a} \cos\frac{\pi y}{b}$$

$$+ \frac{4(1+2\lambda^2)}{(1+8\lambda^2)} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} + \frac{2+\lambda^2}{9(1+2\lambda^2)} \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b} \Big\}$$

$$- \Big\{ \frac{1}{2\lambda^2} \cos\frac{\pi y}{b} + \frac{1}{(1+2\lambda^2)^2} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} - \frac{1}{(1+2\lambda^2)} \left(\frac{\pi x}{a}\right) \sin\left(\frac{\pi x}{a}\right) \cos\frac{\pi y}{b} \Big\}$$

$$- \left(\frac{3}{128} f\pi^2 + \frac{1}{2\lambda^2} \right) + a_1 \cosh\frac{\sqrt{2}\lambda\pi x}{a} \cos\frac{\pi y}{b} \Bigg] - \frac{\sigma}{E}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{a}{R}\right)^2 \left(\frac{f}{8}\right) \lambda^2 \Bigg[\frac{3}{64} f\pi^2 + \frac{1}{16} f\pi^2 \cos\frac{\pi x}{a} + \frac{1}{64} f\pi^2 \cos\frac{2\pi x}{a} - \frac{3}{64} f\pi^2 \cos\frac{2\pi y}{b} - \frac{1}{16} f\pi^2 \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b}$$

$$- \frac{1}{64} f\pi^2 \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b} \Bigg]$$

$$\frac{\sigma_y}{E} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \lambda^2 \Bigg[\left(\frac{3}{128} f\pi^2 - \frac{1}{2\lambda^2} \right) + \left(\frac{1}{16} f\pi^2 \cos\frac{\pi x}{a} + \frac{1}{64} f\pi^2 \cos\frac{2\pi x}{a} \right)$$

$$+ \Big\{ \frac{1}{8(1+2\lambda^2)} f\pi^2 \cos\frac{\pi x}{a} + \frac{1}{16(2+\lambda^2)} f\pi^2 \cos\frac{2\pi x}{a} - \frac{1}{2\lambda^2} - \frac{1}{(1+2\lambda^2)^2} \cos\frac{\pi x}{a} + \frac{1}{(1+2\lambda^2)} \frac{\pi x}{a} \sin\frac{\pi x}{a} + a_1 \cosh\frac{\sqrt{2}\lambda\pi x}{a} \Big\} \cos\frac{\pi y}{b}$$

$$+ \Big\{ -\frac{3}{128} f\pi^2 + \frac{1-4\lambda^2}{16(1+8\lambda^2)} f\pi^2 \cos\frac{\pi x}{a} + \frac{1-\lambda^2}{64(1+2\lambda^2)} f\pi^2 \cos\frac{2\pi x}{a} \Big\} \cos\frac{2\pi y}{b} \Bigg] - \frac{\sigma}{E}$$

476

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cos\frac{\pi x}{a}\, dx = \frac{1}{\pi 8}\int_0^{2\pi} \theta \sin\theta\, d\theta = \frac{1}{8\pi}\Big[\sin\theta - \theta\cos\theta\Big]_0^{2\pi} = -\frac{1}{8\pi}2\pi = -\frac{1}{4}$$

$$\frac{1}{a}\int_0^a \cos\frac{\pi x}{a} \cosh(\sqrt{2}\lambda)\frac{\pi x}{a}\, dx = \frac{1}{2\pi}\int_0^a \left[\cosh\frac{\pi x}{a}(\sqrt{2}\lambda + i) + \cosh\frac{\pi x}{a}(\sqrt{2}\lambda - i)\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{\sqrt{2}\lambda + i}\sinh\frac{\pi x}{a}(\sqrt{2}\lambda + i) + \frac{1}{\sqrt{2}\lambda - i}\sinh\frac{\pi x}{a}(\sqrt{2}\lambda - i)\right] = -\frac{1}{\pi}\frac{\sqrt{2}\lambda}{2\lambda^2+1}\sinh\sqrt{2}\lambda\pi$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cos\frac{2\pi x}{a}\, dx = \frac{1}{2\pi}\int_0^a \left(\frac{\pi x}{a}\right)\left[\sin\frac{3\pi x}{a} - \sin\frac{\pi x}{a}\right] d\left(\frac{\pi}{a}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{9}\left(\sin\theta - \theta\cos\theta\right)_0^{3\pi} - \left(\sin\theta - \theta\cos\theta\right)_0^{\pi}\right] = \frac{1}{2}\left(\frac{1}{3} - 1\right) = -\frac{1}{3}$$

$$\frac{1}{a}\int_0^a \cos\frac{2\pi x}{a} \cosh(\sqrt{2}\lambda)\frac{\pi x}{a}\, dx = \frac{1}{\pi}\frac{\sqrt{2}\lambda \sinh\sqrt{2}\lambda\pi}{9\lambda^2+4}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sin\frac{\pi x}{a}\, dx = 1$$

$$\frac{1}{a}\int_0^a \cosh\frac{\sqrt{2}\lambda\pi x}{a}\, dx = \frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right)^2 \sin^2\frac{\pi x}{a}\, dx = \frac{1}{2\pi}\int_0^a \left\{\left(\frac{\pi x}{a}\right)^2 - \left(\frac{\pi x}{a}\right)^2\cos\frac{2\pi x}{a}\right\} d\left(\frac{\pi x}{a}\right) = \frac{1}{2\pi}\left[\frac{\pi^3}{3} - \frac{1}{8}\left\{2\theta\cos\theta + (\theta^2 - 2)\sin\theta\right\}_0^{2\pi}\right]$$

$$= \frac{\pi^2}{6} - \frac{1}{16\pi}(4\pi) = \frac{\pi^2}{6} - \frac{1}{4}$$

427

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cosh(\sqrt{2}\lambda)\frac{\pi x}{a}\, dx = \frac{1}{2\pi i}\int_0^a \left[\left(\frac{\pi x}{a}\right)\sinh(\sqrt{2}\lambda+i) - \left(\frac{\pi x}{a}\right)\sinh(\sqrt{2}\lambda - i)\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi i}\left[\frac{1}{(\sqrt{2}\lambda+i)^2}\left[\left(\frac{\pi x}{a}\right)(\sqrt{2}\lambda+i)\cosh\frac{\pi x}{a}(\sqrt{2}\lambda+i) - \sinh\frac{\pi x}{a}(\sqrt{2}\lambda+i)\right]\right.$$

$$\left. - \frac{1}{(\sqrt{2}\lambda-i)^2}\left[\left(\frac{\pi x}{a}\right)(\sqrt{2}\lambda-i)\cosh\frac{\pi x}{a}(\sqrt{2}\lambda-i) - \sinh\frac{\pi x}{a}(\sqrt{2}\lambda-i)\right]\right]_0^a$$

$$= \frac{1}{2\pi i}\left[\frac{\pi x}{a}\left\{\frac{\cosh\frac{\pi x}{a}(\sqrt{2}\lambda+i)}{\sqrt{2}\lambda+i} - \frac{\cosh\frac{\pi x}{a}(\sqrt{2}\lambda-i)}{\sqrt{2}\lambda-i}\right\} - \left\{\frac{\sinh\frac{\pi x}{a}(\sqrt{2}\lambda+i)}{(2\lambda^2-1)+2\sqrt{2}\lambda i} - \frac{\sinh\frac{\pi x}{a}(\sqrt{2}\lambda-i)}{(2\lambda^2-1)-2\sqrt{2}\lambda i}\right\}\right]$$

$$= \frac{\cosh\sqrt{2}\lambda\pi}{2\lambda^2+1} - \frac{2\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi}{\pi\left[(2\lambda^2-1)^2+8\lambda^2\right]} = \frac{\cosh\sqrt{2}\lambda\pi}{2\lambda^2+1} - \frac{2\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi}{\pi(2\lambda^2+1)^2}$$

$$\frac{1}{a}\int_0^a \cosh^2(\sqrt{2}\lambda)\frac{\pi x}{a}\, dx = \frac{1}{2a}\int_0^a\left[1+\cosh(2\sqrt{2}\lambda)\frac{\pi x}{a}\right]dx = \frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi$$

$$\frac{1}{gE^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \frac{1}{2}\left(\frac{\sigma}{E}\right)^2 - \left(\frac{a}{R}\right)^2\frac{f}{8}\;\lambda^2\,\frac{\sigma}{E}\left[\left(\frac{3}{128}f\pi^2 - \frac{1}{2\lambda^2}\right)\right]$$

$$+\left(\frac{a}{R}\right)^4\left(\frac{f\pi\lambda^2}{8}\right)^2\left[\frac{1}{2}\left(\frac{3}{128}f\pi^2-\frac{1}{2\lambda^2}\right)^2+\frac{1}{4}\left\{\left(\frac{f\pi^2}{16}\right)^2+\left(\frac{f\pi^2}{64}\right)^2\right\}\right.$$

$$+\frac{1}{8}\left\{2\left(\frac{3}{128}f\pi^2\right)^2+\left(\frac{1-4\lambda^2}{16(1+8\lambda^2)}f\pi^2\right)^2+\left(\frac{1-\lambda^2}{64(1+2\lambda^2)}f\pi^2\right)^2+\left(\frac{f\pi^2}{8(1+2\lambda^2)}-\frac{1}{(1+2\lambda^2)^2}\right)^2+\left(\frac{f\pi^2}{16(2+\lambda^2)}\right)^2+2\left(\frac{1}{2\lambda^2}\right)^2\right\}$$

$$+\frac{1}{4a}\left\{2\int_0^a\left(\frac{f\pi^2}{8(1+2\lambda^2)}-\frac{1}{(1+2\lambda^2)^2}\right)\left(\frac{1}{(1+2\lambda^2)}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right)\cos\frac{\pi x}{a}\,dx\right.$$

$$+2\int_0^a\frac{f\pi^2}{16(2+\lambda^2)}\left(\frac{1}{(1+2\lambda^2)}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right)\cos\frac{2\pi x}{a}\,dx$$

$$-\frac{1}{2\lambda^2}\int_0^a 2\left(\frac{1}{(1+2\lambda^2)}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right)dx$$

$$\left.\left.+\frac{1}{\lambda^4}\int_0^a\left(\frac{1}{(1+2\lambda^2)}\frac{\pi x}{a}\sin\frac{\pi x}{a}+a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right)^2dx\right\}\right]$$

429

$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \frac{1}{2}\left(\frac{\sigma}{E}\right)^2 + \left(\frac{a}{R}\right)^2\frac{f^2}{8}\ \lambda^2\frac{\sigma}{E}\left(\frac{1}{9\lambda^2} - \frac{3}{128}f\pi^2\right)$$

$$+\left(\frac{a}{R}\right)^4\left(\frac{-1}{8}\frac{\lambda^2}{}\right)^2\Big[\frac{f}{8}\left(\frac{3}{64}f\pi^2 - \frac{1}{\lambda^2}\right)^2 + \frac{17}{65536}(f\pi^2)^2 + \frac{(1-4\lambda^2)^2}{2048(1+8\lambda^2)^2}(f\pi^2) + \frac{1}{32768}\frac{(1-\lambda^2)^2}{(1+2\lambda^2)^2}(f\pi^2) + \frac{1}{512}\frac{1}{(1+2\lambda^2)^2}(f\pi^2)$$

$$-\frac{1}{32}\frac{1}{(1+2\lambda^2)^3}(f\pi) + \frac{1}{8(1+2\lambda^2)^4} + \frac{1}{2048}\frac{1}{(2+\lambda^2)^2}(f\pi^2) + \frac{1}{16}\frac{1}{\lambda^4} + \frac{1}{2}\left\{\frac{f\pi^2}{8(1+2\lambda^2)} - \frac{1}{(1+2\lambda^2)^2}\right\}\left\{-\frac{1}{4(1+2\lambda^2)}\right.$$

$$\left.-\frac{1}{(1+2\lambda^2)^2}\right\} + \frac{1}{2}\frac{f\pi^2}{16(2+\lambda^2)}\left\{-\frac{1}{3(1+2\lambda^2)} + \frac{1}{2}\frac{1}{(1+2\lambda^2)(2+\lambda^2)}\right\} - \frac{1}{4\lambda^2}\frac{1}{(1+2\lambda^2)} - \frac{1}{4\lambda^2}\frac{1}{2\lambda^2(1+2\lambda^2)}$$

$$+\frac{1}{4}\frac{1}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6} - \frac{1}{4}\right) + \frac{1}{2}\frac{1}{(1+2\lambda^2)^3}\frac{\pi}{\sqrt{2}\lambda}\coth\sqrt{2}\lambda\pi - \frac{1}{2}\frac{2}{(1+2\lambda^2)^4} + \frac{1}{4}\pi^2\frac{1}{(1+2\lambda^2)^2\lambda^2}\frac{\frac{1}{2}+\frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi}{(\cosh 2\sqrt{2}\lambda\pi - 1)}\Big]$$

$$\frac{\sigma_x}{E} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 = \left(\frac{a}{R}\right)^2\left(\frac{f\pi}{8}\right)\Big[-\left(\frac{x}{a}\right)\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right) + \left(\frac{f\pi}{8}\right)\frac{1}{8}\left(1-\cos\frac{2\pi x}{a}\right)\left(3+4\cos\frac{\pi y}{b}+\cos\frac{2\pi y}{b}\right)\Big]$$

$$=\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\Big[-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a} - \left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{f\pi^2}{64}\Big\{3+4\cos\frac{\pi y}{b}+\cos\frac{2\pi y}{b} - 3\cos\frac{2\pi x}{a} - 4\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b} - \cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}\Big\}\Big]$$

$$=\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\Big[\left\{-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a} + \frac{3}{64}f\pi^2 - \frac{3}{64}f\pi^2\cos\frac{2\pi x}{a}\right\}$$

$$+\left\{-\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a} + \frac{f\pi^2}{16} - \frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\right\}\cos\frac{\pi y}{b}$$

$$+\left\{\frac{f\pi^2}{64} - \frac{f\pi^2}{64}\cos\frac{2\pi x}{a}\right\}\cos\frac{2\pi y}{b}\Big]$$

$$\frac{1}{\gamma E^2 ab}\int_0^a\int_0^b \sigma_x^2\,dy\,dx = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{3}{4}\left(\frac{3}{64}\right)^2(f\pi)^2 + \frac{1}{a}\frac{3}{64}(f\pi)\int_0^a\left(\cos\frac{2\pi x}{a}-1\right)\left(\frac{x}{a}\right)\sin\frac{\pi x}{a}\,dx + \frac{3}{4a}\int_0^a\left(\frac{\pi x}{a}\right)\sin^2\frac{2\pi x}{a}\,dx\right.$$

$$\left. + \frac{3}{8}\left(\frac{f\pi}{16}\right)^2 - \frac{1}{2a}\left(\frac{f\pi}{16}\right)\int_0^a\left(1-\cos\frac{2\pi x}{a}\right)\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\,dx + \frac{1}{8a}\left(\frac{f\pi}{64}\right)^2 3\right].$$

$$= \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{25}{16384}(f\pi)^2 + \frac{3}{64}(f\pi)\left\{-\frac{1}{3}-1\right\} + \frac{3}{4}\left\{\frac{\pi^2}{6}-\frac{1}{4}\right\} - \frac{f\pi^2}{32}\left\{1+\frac{1}{3}\right\} + \frac{3}{16384}(f\pi)^2\right]$$

$$= \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{105}{32768}(f\pi)^2 - \frac{5}{48}(f\pi) + \frac{3}{2}\left(\frac{\pi^2}{12}-\frac{1}{8}\right)\right]$$

$$\frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)^2\left(\frac{f\pi}{8}\lambda\right)\left[-\frac{1}{2}\left(\frac{x}{a}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} + \left(\frac{f\pi}{16}\right)\left(\sin\frac{\pi x}{a}+\frac{1}{2}\sin\frac{2\pi x}{a}\right)\left(\sin\frac{\pi y}{b}+\frac{1}{2}\sin\frac{2\pi y}{b}\right)\right]$$

$$\frac{\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\lambda\left[-\frac{1}{2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi y}{b} - \frac{1}{2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \left(\frac{f\pi^2}{16}\right)\left\{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{2}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}\right.\right.$$

$$\left. + \frac{1}{2}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{1}{4}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}$$

$$+ \frac{1+\lambda^2}{(1+2\lambda^2)^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{2(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{2}\sqrt{2}\lambda a_1\sinh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}$$

$$+ \frac{(f\pi^2)}{64}\left\{-\frac{4}{(1+2\lambda^2)}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} - \frac{4}{(9+\lambda^2)}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b} - \frac{2(1+2\lambda^2)}{(1+8\lambda^2)}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}\right.$$

$$\left.\left. - \frac{2+\lambda^2}{2(1+2\lambda^2)}\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right\}\right]$$

$$\frac{\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)\Bigg[\Big\{-\frac{1}{2}\left(\frac{\pi x}{a}\right) - \frac{1}{2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \frac{f\pi^2}{16}\sin\frac{\pi x}{a} + \frac{f\pi^2}{32}\sin\frac{2\pi x}{a} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\sin\frac{\pi x}{a} + \frac{1}{2(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi x}{a}\right)$$

$$+\frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} - \frac{f\pi^2}{16}\frac{1}{(1+2\lambda^2)}\sin\frac{\pi x}{a} - \frac{f\pi^2}{16}\frac{1}{(2+\lambda^2)}\sin\frac{2\pi x}{a}\Big\}\sin\frac{\pi y}{b}$$

$$+\Big\{\frac{f\pi^2}{32}\sin\frac{\pi x}{a} + \frac{f\pi^2}{64}\sin\frac{2\pi x}{a} - \frac{f\pi^2}{32}\frac{(1+2\lambda^2)}{(1+8\lambda^2)}\sin\frac{\pi x}{a} - \frac{f\pi^2}{128}\frac{2+\lambda^2}{1+2\lambda^2}\sin\frac{2\pi x}{a}\Big\}\sin\frac{2\pi y}{b}\Bigg]$$

$$= \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)\Bigg[\Big\{-\frac{1}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a} + \frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)}\sin\frac{\pi x}{a} + \frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\sin\frac{2\pi x}{a} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\sin\frac{\pi x}{a}$$

$$+\frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a}\Big\}\sin\frac{\pi y}{b}$$

$$+\Big\{\frac{f\pi^2}{16}\frac{3\lambda^2}{(1+8\lambda^2)}\sin\frac{\pi x}{a} + \frac{f\pi^2}{128}\frac{3\lambda^2}{(1+2\lambda^2)}\sin\frac{2\pi x}{a}\Big\}\sin\frac{2\pi y}{b}\Bigg]$$

$$= \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)\Bigg[\Big\{\Big[\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\Big]\sin\frac{\pi x}{a} + \frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\sin\frac{2\pi x}{a} + \frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a}$$

$$-\frac{1}{2}\left(\frac{\pi x}{a}\right) - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\Big\}\sin\frac{\pi y}{b}$$

$$+\Big\{\frac{f\pi^2}{16}\frac{3\lambda^2}{(1+8\lambda^2)}\sin\frac{\pi x}{a} + \frac{f\pi^2}{128}\frac{3\lambda^2}{(1+2\lambda^2)}\sin\frac{2\pi x}{a}\Big\}\sin\frac{2\pi y}{b}\Bigg]$$

$$\frac{1}{E^2 ab}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^2\left(\frac{f\lambda}{8}\right)^2\left[\frac{1}{4}\left\{\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{((1+2\lambda^2)^2}\right\}^2 + \frac{1}{4}\left(\frac{f\pi^2}{32}\right)^2\left(\frac{\lambda^2}{2+\lambda^2}\right)^2\right.$$

$$+\frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2\left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2 + \frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2\left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2 + \frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2\left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2 + \frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2\left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2$$

$$+\left[\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\right]\frac{1}{a}\int_0^a \sin\frac{\pi x}{a}\left\{\frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} - \frac{1}{2}\frac{\pi x}{a} - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\right\}dx$$

$$+\frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\frac{1}{a}\int_0^a \sin\frac{2\pi x}{a}\left\{\frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} - \frac{1}{2}\frac{\pi x}{a} - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\right\}dx$$

$$+\frac{1}{2}\frac{1}{a}\int_0^a\left\{\frac{1}{2}\sqrt{2}\lambda a_1 \sinh\frac{\sqrt{2}\lambda\pi x}{a} - \frac{1}{2}\frac{\pi x}{a} - \frac{\lambda^2}{(1+2\lambda^2)}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\right\}^2 dx$$

$$\frac{1}{a}\int_0^a \sin\frac{\pi x}{a}\sinh(\sqrt{2}\lambda)\left(\frac{\pi x}{a}\right)dx = \frac{1}{2\pi i}\int_0^\pi\left[\cosh\theta(\sqrt{2}\lambda+i) - \cosh(\sqrt{2}\lambda-i)\theta\right]d\theta$$

$$= \frac{1}{2\pi i}\left[\frac{1}{\sqrt{2}\lambda+i}\sinh\theta(\sqrt{2}\lambda+i) - \frac{1}{\sqrt{2}\lambda-i}\sinh\theta(\sqrt{2}\lambda-i)\right]_0^\pi = +\frac{1}{\pi}\frac{1}{(1+2\lambda^2)}\sinh(\sqrt{2}\lambda\pi)$$

$$\frac{1}{a}\int_0^a\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\,dx = \frac{1}{\pi}\Big[\sin\theta - \theta\cos\theta\Big]_0^\pi = 1$$

$$\frac{1}{a}\int_0^a\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi x}{a}\,dx = \frac{1}{8\pi}\Big[\sin\theta - \theta\cos\theta\Big]_0^{2\pi} = -\frac{1}{4}$$

483

$$\frac{\Delta \mathcal{P}}{abtE} = -\left(\frac{a}{R}\right)^2 \frac{(f\lambda)^2}{64}\left(\frac{3}{2}\pi^2 \frac{\sigma}{E}\right)$$

$$\frac{t^2}{12ab}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12ab}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2\lambda^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12ab}\int_0^a\int_0^b\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)\left(\frac{f\pi^2\lambda}{2}\right)^2\frac{1}{4}$$

The part of extensional energy which is independent of a's b's

$$= \left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{64}\Bigg[(f\pi^2)^2\Big\{\frac{1}{32} + \frac{1}{16(1+\lambda^2)^4} + \frac{1}{4(4+\lambda^2)^4} + \frac{1}{64(1+4\lambda^2)^4} + \frac{1}{512} + \frac{1}{32\lambda^4} + \frac{\lambda^4}{16(1+\lambda^2)^4} + \frac{1}{512\lambda^4} +$$

$$+ \frac{\lambda^4}{64(4+\lambda^2)^4} + \frac{\lambda^4}{4(1+4\lambda^2)^4} + \frac{2\lambda^2}{16(1+\lambda^2)^4} + \frac{2\lambda^2}{16(4+\lambda^2)^4} + \frac{2\lambda^2}{16(1+4\lambda^2)^4}\Big\}$$

$$-(f\pi^2)\Big\{\frac{1}{4(1+\lambda^2)^4} + \frac{1}{4\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4}\Big\}$$

$$+\Big\{\frac{1}{4(1+\lambda^2)^4} + \frac{1}{2\lambda^4} + \frac{\lambda^4}{4(1+\lambda^2)^4} + \frac{2\lambda^2}{4(1+\lambda^2)^4}\Big\}\Bigg]$$

$$= \left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{64}\Bigg[(f\pi^2)^2\Big\{\frac{17}{512} + \frac{17}{512\lambda^4} + \frac{1}{16(1+\lambda^2)^2} + \frac{1}{64(4+\lambda^2)^2} + \frac{1}{64(1+4\lambda^2)^2}\Big\}$$

$$-(f\pi^2)\Big\{\frac{1}{4\lambda^4} + \frac{1}{4(1+\lambda^2)^2}\Big\} + \Big\{\frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2}\Big\}\Bigg]$$

The coefficient of $\left\{ a_0 \dfrac{(f\lambda^2)^2}{64} \dfrac{\sinh 2\pi}{2\pi} \right\} \left(\dfrac{a}{R}\right)^4$ in $(A_1 + A_2 + 2A_3)$

$(f\pi^2)$

$$- \frac{1}{4} - \frac{1}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{4(1+\lambda^2)^2} + \frac{\lambda^4}{5(1+4\lambda^2)^2}$$

$$- \frac{2\lambda^2}{4(1+\lambda^2)^2} - \frac{2\lambda^2}{5(1+4\lambda^2)^2}$$

$$= - \frac{1}{4(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)}$$

$$+ \frac{1}{2(1+\lambda^2)^2} - 1 - \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2\lambda^2}{2(1+\lambda^2)^2}$$

$$= \Big(- 1 + \frac{1}{2(1+\lambda^2)^2} + \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}$$

$\left\{ b_2 \dfrac{(f\lambda^2)^2}{64} \dfrac{\sinh 2\pi}{2\pi} \right\} \left(\dfrac{a}{R}\right)^4$

$$(f\pi^2)\Big\{ + \frac{1}{4} - \frac{3}{100(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2\lambda^4}{5(1+4\lambda^2)^2} + \frac{3\lambda^4}{25(1+4\lambda^2)^2}$$

$$- \frac{2\lambda^2}{5(1+4\lambda^2)^2} + \frac{4\lambda^2}{25(1+4\lambda^2)^2} \Big\}$$

$$= (f\pi^2)\left\{ \frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5} \frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2} + \frac{1}{100} \frac{(2\lambda^2+3)(6\lambda^2-1)}{(1+4\lambda^2)^2} \right\}$$

$$\left\{ - 1 - \frac{\lambda^4}{(1+\lambda^2)^2} \right\} =$$

$$\left\{ b_2 \frac{(4\lambda^2)^2}{64} \cosh 2\pi \right\} \left(\frac{a}{R}\right)^4$$

$$(4\pi^2)\left\{ -\frac{1}{4} - \frac{1}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)^2} + \frac{1}{4} + \frac{\lambda^4}{4(1+\lambda^2)} + \frac{\lambda^4}{5(1+4\lambda^2)^2} - \frac{2\lambda^2}{4(1+\lambda^2)^2} - \frac{2\lambda^2}{5(1+4\lambda^2)^2} \right\}$$

$$= (4\pi^2)\left\{ -\frac{1}{4(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} - \frac{1}{20(1+4\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)} \right\}$$

$$\left\{ \frac{1}{2(1+\lambda^2)^2} - 1 - \frac{\lambda^4}{2(1+\lambda^2)} + \frac{2\lambda^2}{2(1+\lambda^2)^2} \right\}$$

$$= \frac{1}{2(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2} - 1$$

$$\left\{ a_4 \frac{(4\lambda^2)^2}{64} \frac{\sinh 4\pi}{4\pi} \right\} \left(\frac{a}{R}\right)^4$$

$$(4\pi^2)\left\{ -\frac{1}{16} - \frac{4}{5}\frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{\lambda^4}{5(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)^2} \right\}$$

$$= (4\pi^2)\left\{ -\frac{1}{5}\frac{1}{(4+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2} \right\}.$$

$$\left\{ b_4 \frac{(4\lambda^2)^2}{64} \frac{\sinh 4\pi}{4\pi} \right\} \left(\frac{a}{R}\right)^4$$

$$(4\pi^2)\left\{ +\frac{1}{16} + \frac{12}{25}\frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{2}{5}\frac{\lambda^4}{(4+\lambda^2)^2} - \frac{3}{25}\frac{\lambda^4}{(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)} + \frac{16\lambda^2}{25(4+\lambda^2)^2} \right\}$$

$$= (4\pi^2)\left\{ \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2}{5}\frac{\lambda^2(1-\lambda^2)}{(4+\lambda^2)^2} \right\}$$

$$\left\{ b_4 \frac{4\lambda^2}{64} \cosh 4\pi \right\} \left(\frac{a}{R}\right)^4$$

$$H\pi^2) \left\{ -\frac{1}{16} - \frac{4}{5} \frac{1}{(4+\lambda^2)^2} + \frac{1}{16} + \frac{1}{5} \frac{\lambda^4}{(4+\lambda^2)^2} - \frac{2\lambda^2}{5(4+\lambda^2)^2} \right\}$$

$$= H\pi^2) \left\{ -\frac{1}{5} \frac{1}{(4+\lambda^2)} - \frac{1}{5} \frac{\lambda^2(1-\lambda^2)}{(4+\lambda^2)^2} \right\}$$

Terms in $A_1 + A_2 + 2A_3$ involving quadratics of a's & b's. $\div \frac{4\lambda^2}{64}\left(\frac{a}{R}\right)^4$

$$\frac{1}{4} \left\{ (1+\lambda^2)^2 \frac{\sinh 4\pi}{4\pi} + (1-\lambda^2)^2 \right\} a_2^2$$

$$+ \frac{1}{4} \left\{ (3\lambda^4 + 2\lambda^2 - 1) \frac{\sinh 4\pi}{4\pi} + (\lambda^2+1)^2 \cosh 4\pi - 4(1-\lambda^2)\lambda^2 \right\} a_2 b_2$$

$$+ \frac{1}{4} \left\{ \left[4\pi^2(\lambda^2+1)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1) \right] \frac{\sinh 4\pi}{4\pi} + \frac{1}{2}(3\lambda^4 + 2\lambda^2 - 1) \cosh 4\pi - \frac{4\pi^2}{3}(1-\lambda^2)^2 + 2(2\lambda^4 - 1) \right\} b_2^2$$

$$\frac{1}{4} \left\{ (1+\lambda^2)^2 \frac{\sinh 8\pi}{8\pi} + (1-\lambda^2)^2 \right\} a_4^2$$

$$+ \frac{1}{4} \left\{ (3\lambda^4 + 2\lambda^2 - 1) \frac{\sinh 8\pi}{8\pi} + (\lambda^2+1)^2 \cosh 8\pi + 4\lambda^2(\lambda^2-1) \right\} a_4 b_4$$

$$+ \frac{1}{4} \left\{ \left[16\pi^2(\lambda^2+1)^2 + \frac{1}{2}(5\lambda^4 + 2\lambda^2 + 1) \right] \frac{\sinh 8\pi}{8\pi} + \frac{1}{2}(3\lambda^4 + 2\lambda^2 - 1) \cosh 8\pi - \frac{16\pi^2}{3}(1-\lambda^2)^2 + 2(2\lambda^4 - 1) \right\} b_4^2$$

---

①	②	③	④	⑤	⑥	⑦	⑧	⑨	
λ	$\sqrt{2}\lambda\pi$	② ÷ 2.3025851	$e^{②}$	$e^{-②}$	$\cosh\sqrt{2}\lambda\pi$	$\sinh\sqrt{2}\lambda\pi$	$\sinh\sqrt{2}\lambda\pi/\sqrt{2}\lambda\pi$	⑦/⑧	$⑧^2$
2.0	8.885768	3.8590400	7228.363	0	3614.182	3614.182	406.7383	8.885768	
1.9	8.441480	3.6660882	4635.411	0	2317.706	2317.706	274.5616	8.441480	
1.8	7.997191	3.4731359	2972.595	0	1486.298	1486.298	185.8525	7.997191	
1.7	7.552903	3.2801841	1906.268	0.001	953.1345	953.1335	126.1943	7.552912	
1.6	7.108614	3.0872318	1222.452	0.001	611.2265	611.2255	85.98378	7.108626	
1.5	6.664326	2.8942800	783.9349	0.0013	391.9681	391.9668	58.81468	6.664460	
1.4	6.220038	2.7013282	502.7223	0.0020	251.3622	251.3602	40.41136	6.220088	
1.3	5.775749	2.5083759	322.3858	0.0031	161.1945	161.1914	27.90831	5.775860	
1.2	5.331461	2.3154241	206.7398	0.0049	103.3724	103.3675	19.38821	5.331714	
1.1	4.887172	2.1224718	132.5781	0.0076	66.29285	66.28525	13.56311	4.887732	
1.0	4.442884	1.9295200	85.01977	0.01176	42.51577	42.50401	9.566761	4.444113	
0.9	3.998596	1.7365682	54.52155	0.01834	27.26995	27.25161	6.815295	4.001287	
0.8	3.554307	1.5436159	34.96358	0.02860	17.49609	17.46749	4.914457	3.560127	
0.7	3.110019	1.3506641	22.42147	0.04460	11.23304	11.18844	3.597547	3.122416	
0.6	2.665730	1.1577118	14.37874	0.06955	7.223995	7.154445	2.683860	2.691644	
0.5	2.221442	0.9647600	9.220617	0.108453	4.664536	4.556083	2.050957	2.274322	
0.4	1.777154	0.7718082	5.913004	0.169119	3.041062	2.871943	1.616035	1.881805	

λ	⑩ λ^4	⑪ λ^2	⑫ $95\lambda^4/65536$	⑬ $1+2\lambda^2$	⑭ $1+8\lambda^2$	⑮ $130+\lambda^2$	⑯ $(1+2\lambda^2)^2$	⑰ $(1+2\lambda^2)/2048(1+$[illegible]$)$	⑱ $(130+\lambda^2)/65536(1+$[illegible]$)$	⑲ $3\lambda^2/$[illegible]
2.0	16.0000	4.00	0.02319336	9.00	33.00	134.00	81.00	0.00001331676	0.0002271864	0.0000361690
1.9	13.0321	3.61	0.01889114	8.22	29.88	133.61	67.5684	0.00001343264	0.0002480203	0.0000391314
1.8	10.4976	3.24	0.01521716	7.48	26.92	133.24	55.9504	0.00001356740	0.0002718023	0.0000424134
1.7	8.3521	2.89	0.01210708	6.78	24.12	132.89	45.9684	0.00001372532	0.0002990768	0.0000460468
1.6	6.5536	2.56	0.0095000	6.12	21.48	132.56	37.4544	0.00001391192	0.0003305074	0.0000500609
1.5	5.0625	2.25	0.00733152	5.50	19.00	132.25	30.2500	0.00001413446	0.0003669045	0.0000544777
1.4	3.8416	1.96	0.00556873	4.92	16.68	131.96	24.0640	0.00001440254	0.0004092581	0.0000596554
1.3	2.8561	1.69	0.00414016	4.38	14.52	131.69	19.1844	0.00001472915	0.0004587740	0.0000645208
1.2	2.0736	1.44	0.00300586	3.88	12.52	131.44	15.0544	0.00001513204	0.0005169111	0.0000700584
1.1	1.4641	1.21	0.00212234	3.4[illegible]	10.68	131.21	11.6964	0.00001563597	0.0005854110	0.0000757615
1.0	1.0000	1.00	0.00144959	[illegible].00	9.00	131.00	9.0000	0.00001627604	0.0006663005	0.0000813802
0.9	0.6561	0.81	0.00095107	2.62	7.48	130.81	6.8644	0.00001710290	0.0007618329	0.0000864259
0.8	0.4096	0.64	0.00059375	2.28	6.12	130.64	5.1984	0.00001795152	0.0008743018	0.0000901720
0.7	0.2401	0.49	0.00034805	1.98	4.92	130.49	3.9204	0.00001965034	0.0010056158	0.0000915434
0.6	0.1296	0.36	0.00018787	1.72	3.88	130.36	2.9584	0.00002164546	0.0011564743	0.0000891265
0.5	0.0625	0.25	0.00009059	1.50	3.00	130.25	2.2500	0.00002441406	0.0013249715	0.0000813802
0.4	0.0256	0.16	0.00003709	1.36	2.28	130.16	1.8496	0.00002912555	0.0014603559	0.0000633583

	(20)	(21)	(22)	(23)	(24)	(25)	(26)	(27)	(28)
λ	λ^4((17)+(18)+(19))	H_1	$3+2\lambda^2$	[illegible]	$(3+2\lambda^2)(1+6\lambda^2)$	$(1+2\lambda^2)^3$	$(4+\lambda^2)$	$(2+\lambda^2)^2$	$5+2\lambda^2$
2.0	0.004426755	0.03082446	11.00	[illegible]	275.0000	729.0000	8.00	36.0000	13.00
1.9	0.003917245	0.02601273	10.22	22.66	231.5852	555.4122	7.61	31.4721	12.22
1.8	0.003440936	0.02186244	9.48	20.[illegible]	193.7712	418.5090	7.24	27.4576	11.48
1.7	0.002997142	0.01830857	8.78	18.34	161.0252	311.6658	6.89	23.9121	10.78
1.6	0.002585265	0.01528961	8.12	16.[illegible]	133.8432	229.2201	6.56	20.7936	10.[illegible]
1.5	0.002204803	0.01274767	7.50	14.50	108.7500	166.3750	6.25	18.0625	9.50
1.4	0.001856707	0.01062978	6.92	12.[illegible]	88.2992	118.3749	5.96	15.6816	8.92
1.3	0.001536650	0.00888116	6.38	11.[illegible]	71.0232	84.03767	5.69	13.6161	8.38
1.2	0.001248518	0.00745872	5.88	9.64	56.6832	58.41107	5.44	11.8336	7.88
1.1	0.000990927	0.00631761	5.42	[illegible]	44.7692	40.00169	5.21	10.3041	7.42
1.0	0.000763957	0.00541789	5.00	[illegible]	35.0000	27.00000	5.00	9.0000	7.00
0.9	0.000567764	0.00472318	4.62	[illegible]	27.0232	17.98473	4.81	7.8961	6.62
0.8	0.000402401	0.00420050	4.28	[illegible]	20.7152	11.65235	4.64	6.9696	6.28
0.7	0.000268146	0.00382054	3.98	3.94	15.6812	7.762392	4.49	6.2001	5.98
0.6	0.000164235	0.00355645	3.72	3.16	11.7552	5.088448	4.36	5.5696	5.72
0.5	0.000089423	0.00338437	3.50	2.50	8.7500	3.37500	4.25	5.0625	5.50
0.4	0.000039753	0.00328121	3.36	1.96	6.5856	2.515456	4.16	4.6656	5.36

λ	(22) $-\frac{3}{64(1+2\lambda^2)}$	(23) $-\frac{1}{384(2+\lambda^2)}$	(24) λ^4((22)+(23))	(25) H_2	(26) $3+2\lambda^4$	(27) $(1+2\lambda^2)^3$	(28) $4+11\lambda^2+10\lambda^4$	(29) $1+3\lambda^2$
2.0	-0.005208333	-0.000447361	-0.09001110	-0.2410528	19.0000	729.000	208.0000	13.00
1.9	-0.005702555	-0.000464201	-0.08036578	-0.2268372	16.0321	555.4122	174.0310	11.83
1.7	-0.006266711	-0.000496978	-0.07100250	-0.2131380	13.4976	418.5090	144.6160	10.72
1.8	-0.006913717	-0.000532549	-0.06219196	-0.2002257	11.3521	311.6658	119.3110	9.67
1.6	-0.007659314	-0.000571089	-0.05393877	-0.1881055	9.5536	229.2209	97.6960	8.68
1.5	-0.008522727	-0.000612745	-0.04624833	-0.1767822	8.0625	166.3750	79.3750	7.75
1.4	-0.009524439	-0.000657618	-0.03912691	-0.1662674	6.8416	118.3949	63.9760	6.88
1.3	-0.010702055	-0.000705736	-0.03258179	-0.1565532	5.8561	84.02767	51.1510	6.07
1.2	-0.012081186	-0.000757025	-0.02662131	-0.1476630	5.0736	58.41107	40.5760	5.32
1.1	-0.013706140	-0.000811267	-0.02125494	-0.1396013	4.4641	40.00169	31.9510	4.63
1.0	-0.015625000	-0.000868056	-0.01649306	-0.1323776	4.0000	27.00000	25.0000	4.00
0.9	-0.017891221	-0.000926750	-0.01234647	-0.1260054	3.6561	17.98473	19.4710	3.43
0.8	-0.020559211	-0.000986427	-0.00882509	-0.1204918	3.4096	11.85235	15.1360	2.92
0.7	-0.023674242	-0.001045850	-0.00593529	-0.1158442	3.2401	7.762392	11.7710	2.47
0.6	-0.027257907	-0.001103461	-0.00367499	-0.1[illegible]20605	3.1296	5.088448	9.2560	2.08
0.5	-0.031250000	-0.001157408	-0.00202546	-0.1091219	3.0625	3.375000	7.3750	1.75
0.4	-0.034466912	-0.001205633	-0.00091322	-0.1069547	3.0256	2.515456	6.0160	1.48

494

H_3

λ	(30) $\frac{3+\lambda^4}{4(1+2\lambda^2)^3}$	(31) $-\frac{3(4+11\lambda^2+10\lambda^4)}{16(1+2\lambda^2)^2}$	(32) $\frac{\pi^2}{24}\frac{1+3\lambda^2}{1+2\lambda^2}$	(33) λ^2((30)+(31)+(32))	(34) $-\frac{9}{16(1+2\lambda^2)^2}$	(35) $\frac{\pi^2}{8}$+(33)+(34)		
2.0	+0.006515775	-0.481481481	+0.594003925	+0.476153068	-0.00785630	+1.7019974		
1.9	+0.007216307	-0.482930075	+0.591836026	+0.419201532	-0.00780827	+1.6450938		
1.8	+0.008062909	-0.484634605	+0.589361408	+0.365438667	-0.00893335	+1.5902059		
1.7	+0.009105988	-0.486656323	+0.586523374	+0.314931938	-0.01026916	+1.5383633		
1.6	+0.010419614	-0.48902415	+0.583252770	+0.267221546	-0.01186213	+1.4896100		
1.5	+0.01211495	-0.49199380	+0.579465415	+0.224069777	-0.01376955	+1.4440008		
1.4	+0.0144467	-0.49848321	+0.575058236	+0.178402561	-0.01615507	+1.3959481		
1.3	+0.01742313	-0.49992676	+0.569905814	+0.147701143	-0.01881692	+1.3625918		
1.2	+0.02171506	-0.505367202	+0.563856270	+0.115493944	-0.02213520	+1.3270593		
1.1	+0.02789945	-0.512192254	+0.556728420	+0.087146369	-0.02611772	+1.2952293		
1.0	+0.03703704	-0.520833333	+0.548311360	+0.064515067	-0.03086190	+1.2673538		
0.9	+0.05082228	-0.53184725	+0.538370601	+0.046449933	-0.03643151	+1.2437190		
0.8	+0.07191823	-0.54593721	+0.526667491	+0.033695046	-0.04280316	+1.2245924		
0.7	+0.10435250	-0.563425237	+0.513003432	+0.026181041	-0.05000794	+1.2098737		
0.6	+0.15376005	-0.586134667	+0.497305652	+0.023195173	-0.05686444	+1.2000314		
0.5	+0.22685165	-0.614583333	+0.47972244	+0.023010237	-0.06317561	+1.1935352		
0.4	+0.30070095	-0.60981592	+0.447516831	+0.022137310	-0.06358824	+1.1922497		

49

λ	(36) $\frac{2}{9}(\frac{1}{\lambda^2}+\lambda^2)+\frac{4}{27}$	(37) $\frac{512}{9}H_1/\lambda^2$	(38) $8H_2/\lambda^2$	(39) H_3/λ^2	(40) (37) (39)	(41) (36) × (39)	(42) $K_0=2\sqrt{(41)}$
2.0	1.0925925	0.4383923	−0.4821056	0.4254993	0.1865356	0.4648973	1.36367
1.9	1.0119277	0.4099267	−0.5026813	0.4557047	0.1868055	0.4611402	1.35814
1.8	0.9367352	0.3838673	−0.5262117	0.4908043	0.1884037	0.4597537	1.35610
1.7	0.8450416	0.3603994	−0.5540565	0.5323056	0.1918426	0.4498204	1.34138
1.6	0.8038425	0.3397691	−0.5878277	0.5816789	0.1977045	0.4677390	1.36782
1.5	0.7469135	0.3223115	−0.6285551	0.6417781	0.2068525	0.4793527	1.38470
1.4	0.6970823	0.3085288	−0.6786200	0.7122184	0.2197377	0.4964748	1.40922
1.3	0.6551961	0.2989582	−0.7410802	0.8062173	0.2410402	0.5282632	1.45364
1.2	0.6224691	0.2946655	−0.8203500	0.9215190	0.2715546	0.5736482	1.51478
1.1	0.6006917	0.2970263	−0.9229838	1.0704374	0.3179481	0.6430029	1.60374
1.0	0.5925925	0.3082177	−1.0590288	1.2673538	0.3906209	0.7510244	1.73324
0.9	0.6024965	0.3317240	−1.2444778	1.5354556	0.5093475	0.9251066	1.92364
0.8	0.6375925	0.3733778	−1.5061475	1.9134256	0.7144306	1.2197858	2.20908
0.7	0.7105517	0.4435638	−1.8913339	2.4671300	1.0952167	1.754445	2.64910
0.6	0.8454320	0.5620069	−2.4902333	3.3334206	1.8734054	2.8181804	3.35748
0.5	1.0925925	0.7701322	−3.4919008	4.7741408	3.6767196	5.2161904	4.56778
0.4	1.5725925	1.1666524	−5.3477450	7.4515606	8.6933811	11.7182683	6.84638

C A B

496

$$K = 2\left\{ A\left(\frac{\delta}{t}\right)^2 + B \right\}^{\frac{1}{2}} - C\left(\frac{\delta}{t}\right)$$

$$2A\left(\frac{\delta}{t}\right) = C\left\{ A\left(\frac{\delta}{t}\right)^2 + B \right\}^{\frac{1}{2}}$$

$$(4A^2 - C^2A)\left(\frac{\delta}{t}\right)^2 = C^2B$$

$$\left(\frac{\delta}{t}\right)^2 = \frac{C^2B}{4A^2 - C^2A}$$

$$\boxed{\left(\frac{\delta}{t}\right) = \sqrt{\frac{B}{\left(\frac{2A}{C}\right)^2 - A}}}$$

$$K_{min} = \left(\frac{4A}{C} - C\right)\sqrt{\frac{C^2B}{4A^2 - C^2A}}$$

$$\boxed{K_{min.} = \left[2\left(\frac{2A}{C}\right) - C\right]\sqrt{\frac{B}{\left(\frac{2A}{C}\right)^2 - A}}}$$

	(43)	(44)	(45)	(46)	(47)	(48)	(49)	(50)
λ	$2A/C$	$(43)^2 - A$	$(41) \div (44)$	$? = (45)^{\frac{1}{2}}$	$2(43) - C$	$K_{min.}$		
2.0	0.7238371	0.4122883	1.127602					
1.9	0.7432289	0.3655837	1.261381					
1.8	0.7160008	0.3242534	1.417884					
1.7	0.6922496	0.2873669	1.565317					
1.6	0.6726591	0.2547658	1.835957					
1.5	0.6581802	0.2263487	2.117782				Mistake made !!!	
1.4	0.6476062	0.1996539	2.486177					
1.3	0.6505104	0.1821236	2.900575					
1.2	0.6620457	0.1667499	3.440171					
1.1	0.6889571	0.1567138	4.103039					
1.0	0.7376965	0.1535752	4.890271	2.211396	0.9207461			
0.9	0.8185591	0.1606915	5.757035					
0.8	0.9486861	0.1855747	6.574095					
0.7	1.1581421	0.2460764	7.129676					
0.6	1.5046023	0.3904227	7.218080					
0.5	2.1058557	0.7579086	6.882348					
0.4	3.2512325	1.8771317	1.242646					

451

$$\frac{w}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1 - \left(\frac{x}{a}\right)^2 - \frac{f}{4}\left(1+\cos\frac{2\pi x}{a}\right)\left(1+\cos\frac{2\pi y}{a}\right)\right]$$

$$\frac{w_0}{R} = \frac{1}{2}\left(\frac{a}{R}\right)^2\left[1 - \left(\frac{x}{a}\right)^2\right]$$

$$R\frac{\partial^2 w}{\partial x^2} = \left[-1 + \frac{f}{2}\pi^2 \cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\right]$$

$$R\frac{\partial^2 w}{\partial y^2} = \left[+\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a}\right]$$

$$R\frac{\partial^2 w}{\partial x \partial y} = \left[-\frac{f}{2}\pi^2 \sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a}\right]$$

$$R\frac{\partial^2 w_0}{\partial x^2} = [-1], \qquad R\frac{\partial^2 w_0}{\partial y^2} = R\frac{\partial^2 w_0}{\partial x \partial y} = 0$$

Thus

$$\nabla^4 F = \frac{E}{R^2}\left[\frac{f^2}{4}\pi^4 \sin^2\frac{2\pi x}{a}\sin^2\frac{2\pi y}{a}\right.$$

$$\left. + \frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} - \frac{f^2}{4}\pi^4\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\cos\frac{2\pi y}{a}\right]$$

$$= \frac{E}{R^2}\left[\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} + \frac{f^2}{4}\pi^4\left\{\frac{1}{4}\left(1-\cos\frac{4\pi x}{a}\right)\left(1-\cos\frac{4\pi y}{a}\right)\right.\right.$$

$$\left.\left. - \frac{1}{4}\left(2\cos\frac{2\pi x}{a} + 1 + \cos\frac{4\pi x}{a}\right)\left(2\cos\frac{2\pi y}{a} + 1 + \cos\frac{4\pi y}{a}\right)\right\}\right]$$

$$\nabla^4 F = \frac{E}{R^2}\Big[\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} + \frac{f^2}{16}\pi^4\Big\{ \cancel{X} - 2\cos\frac{4\pi x}{a} - 2\cos\frac{4\pi y}{a} + \cancel{\cos\frac{4\pi x}{a}\cos\frac{4\pi y}{a}}$$

$$- 4\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a} - 2\cos\frac{2\pi y}{a} - 2\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} - 2\cos\frac{2\pi x}{a} - \cancel{X} - \cancel{\cos\frac{4\pi x}{a}}$$

$$- 2\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a} - \cancel{\cos\frac{4\pi y}{a}} - \cancel{\cos\frac{4\pi x}{a}\cos\frac{4\pi y}{a}} \Big\}\Big]$$

$$= \frac{E}{R^2}\Big[\frac{f}{2}\pi^2\left(1+\cos\frac{2\pi x}{a}\right)\cos\frac{2\pi y}{a} - \frac{f^2}{8}\pi^4\Big\{\cos\frac{2\pi x}{a} + \cos\frac{2\pi y}{a} + 2\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}$$

$$+ \cos\frac{4\pi x}{a} + \cos\frac{4\pi y}{a} + \cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\Big\}\Big]$$

$$\nabla^4 F = \frac{E}{R^2}\frac{(f\pi^2)}{2}\Big[-\frac{(f\pi^2)}{4}\cos\frac{2\pi x}{a} + \left\{1 - \frac{(f\pi^2)}{4}\right\}\cos\frac{2\pi y}{a} + \left\{1 - \frac{f\pi^2}{2}\right\}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}$$

$$- \frac{f\pi^2}{4}\Big\{\cos\frac{4\pi x}{a} + \cos\frac{4\pi y}{a} + \cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\Big\}\Big]$$

$$\nabla^4 F = \frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = C\cos\frac{2\pi x}{a}, \qquad \text{Put } F = c\cos\frac{2\pi x}{a}$$

$$c\left\{\left(\frac{2\pi}{a}\right)^4 - \cancel{2\left(\frac{2\pi}{a}\right)^2}\right\} = C \qquad c = \frac{C}{\left(\frac{2\pi}{a}\right)^4 - \cancel{2\left(\frac{2\pi}{a}\right)^2}}$$

452

Therefore the particular integral is

$$E\left(\frac{a}{R}\right)^2 \frac{f}{8}\left[-\frac{f\pi^2}{4}\frac{\cos\frac{2\pi x}{a}}{\left(\frac{2\pi}{a}\right)^2}+\left(1-\frac{f\pi^2}{4}\right)\frac{\cos\frac{2\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2}+\frac{1}{4}\left(1-\frac{f\pi^2}{2}\right)\frac{\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2}\right.$$

$$\left.-\frac{f\pi^2}{4}\left\{\frac{1}{4}\frac{\cos\frac{4\pi x}{a}}{\left(\frac{4\pi}{a}\right)^2}+\frac{1}{4}\frac{\cos\frac{4\pi y}{a}}{\left(\frac{4\pi}{a}\right)^2}+\frac{1}{25}\frac{\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2}+\frac{1}{25}\frac{\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}}{\left(\frac{2\pi}{a}\right)^2}\right\}\right]$$

The complementary function can be taken as

$$F = \frac{E}{\left(\frac{2\pi}{a}\right)^2}\left[A_2\cosh\left(\frac{2\pi y}{a}\right)+B_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right]\cos\frac{2\pi x}{a}$$

$$+\frac{E}{\left(\frac{4\pi}{a}\right)^2}\left[A_4\cosh\left(\frac{4\pi y}{a}\right)+B_4\left(\frac{4\pi y}{a}\right)\sinh\left(\frac{4\pi y}{a}\right)\right]\cos\frac{4\pi x}{a}$$

$$+ E a^2\left(\frac{x}{a}\right)^2 C$$

$$\frac{\sigma_y}{E} = -\left[A_2 \cosh\left(\frac{2\pi y}{a}\right) + B_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right]\cos\frac{2\pi x}{a} - \left[A_4 \cosh\left(\frac{4\pi y}{a}\right) + B_4\left(\frac{4\pi y}{a}\right)\sinh\left(\frac{4\pi y}{a}\right)\right]\cos\frac{4\pi x}{a}$$

$$+ 2C + \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\frac{f\pi^2}{4}\cos\frac{2\pi x}{a} + \frac{f}{4}\left(\frac{f\pi^2}{2} - 1\right)\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a}\right.$$

$$\left. + \frac{f\pi^2}{4}\left\{\frac{f}{4}\cos\frac{4\pi x}{a} + \frac{4}{25}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a} + \frac{1}{25}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\right\}\right]$$

The condition is $\frac{\sigma_y}{E} = -\frac{\sigma}{E}$, when $y = \pm a$.

Thus $-\frac{\sigma}{E} = 2C$ $\qquad \boxed{C = -\frac{\sigma}{2E}}$

$A_2 \cosh 2\pi + 2\pi B_2 \sinh 2\pi = \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\frac{f\pi^2}{4} + \frac{f}{4}\left(\frac{f\pi^2}{2} - 1\right) + \frac{f\pi^2}{4}\left(\frac{1}{25}\right)\right]$	
$A_4 \cosh 4\pi + 4\pi B_4 \sinh 4\pi = \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\frac{f\pi^2}{4}\left(\frac{f}{4} + \frac{4}{25}\right)\right]$	
$(A_2 + B_2)\sinh 2\pi + 2\pi B_2 \cosh 2\pi = 0$	from shear stress consideration
$(A_4 + B_4)\sinh 4\pi + 4\pi B_4 \cosh$ [illegible] $= 0$	

$$\frac{\sigma_y}{E} = -\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\left\{\frac{f\pi^2}{4} + \frac{1}{4}\left(\frac{f\pi^2}{2} - 1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{100}\cos\frac{4\pi y}{a}\right\} - \left\{a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\right]\cos\frac{2\pi x}{a}$$

$$+ \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\left\{\frac{f\pi^2}{16} + \frac{f\pi^2}{25}\cos\frac{2\pi y}{a}\right\} - \left\{a_4 \cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\right]\cos\frac{4\pi x}{a}$$

$$\frac{1}{aE^2}\int_0^a \sigma_y^2\,dy = \left(\frac{\sigma}{E}\right)^2 + \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\left[\left\{\frac{f\pi^2}{4} + \frac{1}{4}\left(\frac{f\pi^2}{2} - 1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{100}\cos\frac{4\pi y}{a}\right\} - \left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\right]^2$$

$$+ \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\left[\left\{\frac{f\pi^2}{16} + \frac{f\pi^2}{25}\cos\frac{2\pi y}{a}\right\} - \left\{a_4\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\right]^2$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_y^2\,dx\,dy = \left(\frac{\sigma}{E}\right)^2 + \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\left[2\left(\frac{f\pi^2}{4}\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2} - 1\right)^2 + \left(\frac{f\pi^2}{100}\right)^2 + 2\left(\frac{f\pi^2}{16}\right)^2 + \left(\frac{f\pi^2}{25}\right)^2\right]$$

$$-\left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[\int_0^1 \left\{\frac{f\pi^2}{4} + \frac{1}{4}\left(\frac{f\pi^2}{2} - 1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{100}\cos\frac{4\pi y}{a}\right\}\left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)\right.$$

$$\left. + \int_0^1 \left\{\frac{f\pi^2}{16} + \frac{f\pi^2}{25}\cos\frac{2\pi y}{a}\right\}\left\{a_4\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\} d\left(\frac{y}{a}\right)\right]$$

$$+ \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\left[\int_0^1 \left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}^2 d\left(\frac{y}{a}\right) + \int_0^1 \left\{a_4\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right)\right]$$

45b

$$\int_0^1 \left\{a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right) = \frac{1}{2\pi} \int_0^1 \left\{a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{2\pi y}{a}\right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left\{a_2 \cosh u + b_2 u \sinh u\right\} du = \frac{1}{2\pi} \Big[a_2 \sinh u + b_2 (u \cosh u - \sinh u)\Big]_0^{2\pi}$$

$$= \frac{1}{2\pi} \Big[(a_2 - b_2) \sinh 2\pi + b_2\, 2\pi \cosh 2\pi \Big]$$

$$\int_0^1 \cos\frac{2\pi y}{a} \left\{ a_2 \cosh\left(\frac{2\pi y}{a}\right) + b_2 \left(\frac{2\pi y}{a}\right) \sinh\left(\frac{2\pi y}{a}\right)\right\} d\left(\frac{y}{a}\right)$$

$$= a_2 \int_0^1 \frac{1}{2} \left\{ \cosh \frac{2\pi}{a}(y+iy) + \cosh\frac{2\pi}{a}(y-iy)\right\} d\left(\frac{y}{a}\right) + b_2 \int_0^1 \frac{1}{2}\left(\frac{2\pi y}{a}\right)\left\{ \sinh\frac{2\pi}{a}(y+iy) + \sinh\frac{2\pi}{a}(y-iy)\right\} d\left(\frac{y}{a}\right)$$

$$= \frac{a_2}{2}\frac{1}{2\pi}\Big[\frac{1}{1+i} \sinh\frac{2\pi}{a}(y+iy) + \frac{1}{1-i}\sinh\frac{2\pi}{a}(y-iy)\Big]_0^1$$

$$+ \frac{b_2}{2}\frac{1}{2\pi}\Big[\frac{1}{(1+i)^2}\left\{ \frac{2\pi y}{a}(1+i)\cosh\frac{2\pi}{a}(y+iy) - \sinh\frac{2\pi}{a}(y+iy)\right\}$$

$$+ \frac{1}{(1-i)^2}\left\{ \frac{2\pi y}{a}(1-i)\cosh\frac{2\pi}{a}(y-iy) - \sinh\frac{2\pi}{a}(y-iy)\right\}\Big]_0^1$$

$$\int_0^1 \cos\frac{2\pi y}{a}\left\{a_2\cosh\frac{2\pi y}{a}+b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}d\left(\frac{y}{a}\right)$$

$$=\frac{a_2}{2}\frac{1}{2\pi}\left[\sinh\frac{2\pi y}{a}\cos\frac{2\pi y}{a}+\cosh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1$$

$$+\frac{b_2}{2}\frac{1}{2\pi}\left[\left(\frac{2\pi y}{a}\right)\left\{\cosh\frac{2\pi y}{a}\cos\frac{2\pi y}{a}+\sinh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right\}-\cosh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1$$

$$=\frac{a_2}{4\pi}\left[\sinh 2\pi\right]+\frac{b_2}{4\pi}\left[2\pi\cosh 2\pi\right]$$

15

$$\int_0^1 \cos\frac{4\pi y}{a}\left\{a_2\cosh\frac{2\pi y}{a}+b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}d\left(\frac{y}{a}\right)$$

$$=\frac{a_2}{4\pi}\left[\frac{1}{1+2i}\sinh\frac{2\pi y}{a}(1+2i)+\frac{1}{1-2i}\sinh\frac{2\pi y}{a}(1-2i)\right]_0^1$$

$$+\frac{b_2}{4\pi}\left[\frac{1}{(1+2i)^2}\left\{\frac{2\pi y}{a}(1+2i)\cosh\frac{2\pi y}{a}(1+2i)-\sinh\frac{2\pi y}{a}(1+2i)\right\}\right.$$

1-4

$$\left.+\frac{1}{(1-2i)^2}\left\{\frac{2\pi y}{a}(1-2i)\cosh\frac{2\pi y}{a}(1-2i)-\sinh\frac{2\pi y}{a}(1-2i)\right\}\right]_0^1$$

$$-\frac{a_2}{4\pi}\left[\frac{2}{5}\sinh\frac{2\pi y}{a}\cos\frac{4\pi y}{a}+\frac{4}{5}\cosh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right]_0^1$$

$$+\frac{b_2}{4\pi}\left[\left(\frac{2\pi y}{a}\right)\left\{\frac{2}{5}\cosh\frac{2\pi y}{a}\cos\frac{4\pi y}{a}+\frac{4}{5}\sinh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right\}+\left\{\frac{6}{25}\sinh\frac{2\pi y}{a}\cos\frac{4\pi y}{a}-\frac{8}{25}\cosh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right\}\right]_0$$

48

$$\int_0^1 \cos\frac{4\pi y}{a}\left\{a_2\cosh\frac{2\pi y}{a}+b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}d\left(\frac{y}{a}\right)$$

$$=\frac{a_2}{4\pi}\,\frac{2}{5}\sinh 2\pi+\frac{b_2}{4\pi}\left[2\pi\,\frac{2}{5}\cosh 2\pi+\frac{6}{25}\sinh 2\pi\right]$$

$$\int_0^1\left\{a_4\cosh\frac{4\pi y}{a}+b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}d\left(\frac{y}{a}\right)=\frac{1}{4\pi}\left[(a_4-b_4)\sinh 4\pi+b_2\,4\pi\cosh 4\pi\right]$$

$$\int_0^1\cos\frac{2\pi y}{a}\left\{a_4\cosh\left(\frac{4\pi y}{a}\right)+b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}d\left(\frac{y}{a}\right)$$

$$=\frac{a_4}{2}\int_0^1\left[\cosh\frac{2\pi y}{a}(2+i)+\cosh\frac{2\pi y}{a}(2-i)\right]d\left(\frac{y}{a}\right)+\frac{b_4}{2}\int_0^1\left(\frac{4\pi y}{a}\right)\left[\sinh\frac{2\pi y}{a}(2+i)+\sinh\frac{2\pi y}{a}(2-i)\right]d\left(\frac{y}{a}\right)$$

$$=\frac{a_4}{2}\,\frac{1}{2\pi}\left[\frac{1}{2+i}\sinh\frac{2\pi y}{a}(2+i)+\frac{1}{2-i}\sinh\frac{2\pi y}{a}(2-i)\right]_0^1+\frac{b_4}{2\pi}\left[\frac{1}{(2+i)^2}\left\{\frac{2\pi y}{a}(2+i)\cosh\frac{2\pi y}{a}(2+i)-\sinh\frac{2\pi y}{a}(2+i)\right\}\right.$$

$$\left.+\frac{1}{(2-i)^2}\left\{\frac{2\pi y}{a}(2-i)\cosh\frac{2\pi y}{a}(2-i)-\sinh\frac{2\pi y}{a}(2-i)\right\}\right]_0^1$$

$$=\frac{a_4}{4\pi}\left[\frac{4}{5}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a}+\frac{2}{5}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1+\frac{b_4}{2\pi}\left[\left(\frac{2\pi y}{a}\right)\frac{4}{5}\cosh\frac{4\pi y}{a}\cos\frac{2\pi y}{a}\right.$$

$$\left.+\left(\frac{2\pi y}{a}\right)\frac{2}{5}\sinh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}-\left\{\frac{6}{25}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a}+\frac{8}{25}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right\}\right]_0^1$$

459

$$\int_0^1 \cos\frac{2\pi y}{a}\left\{a_4\cosh\left(\frac{4\pi y}{a}\right)+b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}d\left(\frac{y}{a}\right)$$

$$=\frac{a_4}{4\pi}\frac{4}{5}\sinh 4\pi+\frac{b_4}{2\pi}\left[2\pi\frac{4}{5}\cosh 4\pi-\frac{6}{25}\sinh 4\pi\right]$$

$$\int_0^1\left\{a_2\cosh\left(\frac{2\pi y}{a}\right)+b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}^2 d\left(\frac{y}{a}\right)$$

$$=\frac{a_2^2}{2}\int_0^1\left(\cosh\frac{4\pi y}{a}+1\right)d\left(\frac{y}{a}\right)+a_2b_2\int_0^1\left(\frac{2\pi y}{a}\right)\sinh\frac{4\pi y}{a}\,d\left(\frac{y}{a}\right)+\frac{b_2^2}{2}\int_0^1\left(\frac{2\pi y}{a}\right)^2\left[\cosh\frac{4\pi y}{a}-1\right]d\left(\frac{y}{a}\right)$$

$$=\frac{a_2^2}{2}\left[\frac{1}{4\pi}\sinh\frac{4\pi y}{a}+\frac{y}{a}\right]_0^1+\frac{a_2b_2}{8\pi}\left[\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a}-\sinh\frac{4\pi y}{a}\right]_0^1$$

$$+\frac{b_2^2}{2}\left[\frac{1}{16\pi}\left\{\left(\left(\frac{4\pi y}{a}\right)^2+2\right)\sinh\frac{4\pi y}{a}-\frac{8\pi y}{a}\cosh\frac{4\pi y}{a}\right\}-\frac{4\pi^2}{3}\left(\frac{y}{a}\right)^3\right]_0^1$$

$$=\frac{a_2^2}{2}\left[\frac{1}{4\pi}\sinh 4\pi+1\right]+\frac{a_2b_2}{8\pi}\left[4\pi\cosh 4\pi-\sinh 4\pi\right]$$

$$+\frac{b_2^2}{2}\left[\frac{1}{16\pi}\left\{(16\pi^2+2)\sinh 4\pi-8\pi\cosh 4\pi\right\}-\frac{4\pi^2}{3}\right]$$

$$\int_0^1 \left\{ a_4 \cosh\left(\frac{4\pi y}{a}\right) + b_4 \left(\frac{4\pi y}{a}\right) \sinh\left(\frac{4\pi y}{a}\right) \right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{a_4^2}{2}\left[\frac{1}{8\pi}\sinh 8\pi + 1\right] + \frac{a_2 b_2}{16\pi}\left[8\pi \cosh 8\pi - \sinh 8\pi\right] + \frac{b_2^2}{2}\left[\frac{1}{32\pi}\left\{(64\pi^2+2)\sinh 8\pi - 16\pi\cosh 8\pi\right\} - \frac{16\pi^2}{3}\right]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\left[\frac{1}{8}(f\pi^2)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f\pi^2)^2}{10{,}000} + \frac{1}{128}(f\pi^2)^2 + \frac{1}{625}(f\pi^2)^2\right]$$

$$- \left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[\frac{f\pi^2}{4}\left\{\frac{(a_2-b_2)}{2\pi}\sinh 2\pi + b_2\cosh 2\pi\right\} + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\left\{a_2\frac{\sinh 2\pi}{4\pi} + b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+ \frac{f\pi^2}{100}\left\{\frac{1}{10\pi}a_2\sinh 2\pi + b_2\left(\frac{1}{5}\cosh 2\pi + \frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left. + \frac{f\pi^2}{16}\left\{\frac{(a_4-b_4)}{4\pi}\sinh 4\pi + b_2\cosh 4\pi\right\} + \frac{f\pi^2}{25}\left\{\frac{1}{5\pi}a_4\sinh 4\pi + b_4\left(\frac{4}{5}\cosh 4\pi - \frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+ \left(\frac{a}{R}\right)^2\frac{f^2}{128}\left[\frac{a_2^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{a_2 b_2}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right.$$

$$\left. + \frac{a_4^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right) + \frac{a_2 b_2}{2}\left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right) + \frac{b_2^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{1}{2}\cosh 8\pi - \frac{16\pi^2}{3}\right)\right]$$

$$\frac{1}{E}\sigma_x = \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[\left\{(a_2+2b_2)\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\cos\frac{2\pi x}{a}\right.$$

$$+\left\{(a_4+2b_4)\cosh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\sinh\left(\frac{4\pi y}{a}\right)\right\}\cos\frac{4\pi x}{a}$$

$$+\left(\frac{f\pi^2}{4}-1\right)\cos\frac{2\pi y}{a} + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{a} + \frac{f\pi^2}{4}\left\{\frac{1}{4}\cos\frac{4\pi y}{a} + \frac{1}{25}\cos\frac{4\pi x}{a}\cos\frac{2\pi y}{a}\right.$$

$$\left.\left. + \frac{4}{25}\cos\frac{2\pi x}{a}\cos\frac{4\pi y}{a}\right\}\right]$$

$$\frac{\sigma_x}{E} = \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\left\{\left(\frac{f\pi^2}{4}-1\right) + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{25}\cos\frac{4\pi y}{a} + (a_2+2b_2)\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right\}\cos\frac{2\pi x}{a}\right.$$

$$\left. + \left\{\frac{1}{16}f\pi^2 + \frac{f\pi^2}{100}\cos\frac{2\pi y}{a} + (a_4+2b_4)\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\cos\frac{4\pi x}{a}\right]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_x^2\,dx\,dy = \left(\frac{a}{R}\right)^4\frac{f^2}{256}\left[2\left(\frac{f\pi^2}{4}-1\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + 2\left(\frac{f\pi^2}{16}\right)^2 + \frac{(f\pi^2)^2}{10{,}000} + \left(\frac{f\pi^2}{25}\right)^2\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{64}\left[\int_0^1\left\{\left(\frac{f\pi^2}{4}-1\right)+\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{f\pi^2}{25}\cos\frac{4\pi y}{a}\right\}\left\{(a_2+2b_2)\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right\}d\left(\frac{y}{a}\right)\right.$$

$$\left. + \int_0^1\left\{\frac{1}{16}f\pi^2 + \frac{f\pi^2}{100}\right\}\left\{(a_4+2b_4)\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}d\left(\frac{y}{a}\right)\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\int_0^1\left\{(a_2+2b_2)\cosh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\sinh\frac{2\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right) + \int_0^1\left\{(a_4+2b_4)\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right)\right]$$

462

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \sigma_x^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\left[2\left(\frac{f\pi^2}{4}-1\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{1}{128}(f\pi^2)^2 + \frac{(f\pi^2)^2}{10{,}000} + \left(\frac{f\pi^2}{25}\right)^2\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{64}\left[\left(\frac{f\pi^2}{4}-1\right)\left\{\frac{(a_2+b_2)}{2\pi}\sinh 2\pi + b_2\cosh 2\pi\right\} + \frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\left\{(a_2+2b_2)\frac{\sinh 2\pi}{4\pi} + b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+\frac{f\pi^2}{25}\left\{\frac{1}{10\pi}(a_2+2b_2)\sinh 2\pi + b_2\left(\frac{1}{5}\cosh 2\pi + \frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\pi^2}{16}\left\{\frac{(a_4+b_4)}{4\pi}\sinh 4\pi + b_2\cosh 4\pi\right\} + \frac{f\pi^2}{100}\left\{\frac{1}{5\pi}(a_4+2b_4)\sinh 4\pi + b_4\left(\frac{4}{5}\cosh 4\pi - \frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^2\frac{f^2}{128}\left[\frac{(a_2+2b_2)^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{(a_2+2b_2)b_2}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right.$$

$$\left.+\frac{(a_4+2b_4)^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right) + \frac{(a_4+2b_4)b_4}{2}\left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right) + \frac{b_4^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{1}{2}\cosh 8\pi - \frac{16\pi^2}{3}\right)\right]$$

463

$$\frac{1}{E}\tau_{xy} = \left(\frac{a}{R}\right)^2 \frac{f}{8}\Big[\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} + \frac{f\pi^2}{4}\Big\{\frac{2}{25}\sin\frac{4\pi x}{a}\sin\frac{2\pi y}{a} + \frac{2}{25}\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a}\Big\}$$

$$+ \Big\{(a_2+b_2)\sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\Big\}\sin\frac{2\pi x}{a} + \Big\{(a_4+b_4)\sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a}\Big\}\sin\frac{4\pi x}{a}\Big]$$

$$\frac{1}{aE^2}\int_0^a \tau_{xy}^2\,dx = \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\Big[\Big\{\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\sin\frac{2\pi y}{a} + (a_2+b_2)\sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\Big\}^2$$

$$\frac{f\pi^2}{50}\sin\frac{4\pi y}{a}$$

$$+ \Big\{\frac{f\pi^2}{50}\sin\frac{2\pi y}{a} + (a_4+b_4)\sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a}\Big\}^2\Big]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{f^2}{256}\Big[\frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + 2\left(\frac{f\pi^2}{50}\right)^2\Big]$$

$$+ \left(\frac{a}{R}\right)^4 \frac{f^2}{64}\Big[\int_0^1\Big\{\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\sin\frac{2\pi y}{a} + \frac{f\pi^2}{50}\sin\frac{4\pi y}{a}\Big\}\Big\{(a_2+b_2)\sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\Big\}\,d\left(\frac{y}{a}\right)$$

$$+ \frac{1}{50}f\pi^2\int_0^1 \sin\frac{2\pi y}{a}\Big\{(a_4+b_4)\sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a}\Big\}\,d\left(\frac{y}{a}\right)\Big]$$

$$+ \left(\frac{a}{R}\right)^4 \frac{f^2}{128}\Big[\int_0^1\Big\{(a_2+b_2)\sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\Big\}^2 d\left(\frac{y}{a}\right) + \int_0^1\Big\{(a_4+b_4)\sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a}\Big\}^2 d\left(\frac{y}{a}\right)\Big]$$

$$\int_0^1 \sin\frac{2\pi y}{a}\sinh\frac{2\pi y}{a}\, d\left(\frac{y}{a}\right) = \frac{1}{2i}\int_0^1\left[\cosh\frac{2\pi y}{a}(1+i) - \cosh\frac{2\pi y}{a}(1-i)\right] d\left(\frac{y}{a}\right)$$

$$= \frac{1}{4\pi i}\left[\frac{1}{1+i}\sinh\frac{2\pi y}{a}(1+i) - \frac{1}{1-i}\sinh\frac{2\pi y}{a}(1-i)\right]_0^1 = \frac{1}{4\pi}\left[-\sinh\frac{2\pi y}{a}\cos\frac{2\pi y}{a} + \cosh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1$$

$$= -\frac{\sinh 2\pi}{4\pi}$$

$$\int_0^1 \sin\frac{2\pi y}{a}\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\, d\left(\frac{y}{a}\right) = \frac{1}{2i}\int_0^1\left(\frac{2\pi y}{a}\right)\left[\sinh\frac{2\pi y}{a}(1+i) - \sinh\frac{2\pi y}{a}(1-i)\right] d\left(\frac{y}{a}\right)$$

$$= \frac{1}{4\pi i}\left[\frac{1}{(1+i)^2}\left\{\frac{2\pi y}{a}(1+i)\cosh\frac{2\pi y}{a}(1+i) - \sinh\frac{2\pi y}{a}(1+i)\right\} - \frac{1}{(1-i)^2}\left\{\frac{2\pi y}{a}(1-i)\cosh\frac{2\pi y}{a}(1-i) - \sinh\frac{2\pi y}{a}(1-i)\right\}\right]_0^1$$

$$= \frac{1}{4\pi}\left[\frac{2\pi y}{a}\left\{-\cosh\frac{2\pi y}{a}\cos\frac{2\pi y}{a} + \sinh\frac{2\pi y}{a}\sin\frac{2\pi y}{a}\right\} + \sinh\frac{2\pi y}{a}\cos\frac{2\pi y}{a}\right]_0^1$$

$$= \left[-\frac{1}{2}\cosh 2\pi + \frac{\sinh 2\pi}{4\pi}\right]$$

$$\int_0^1 \sin\frac{4\pi y}{a}\sinh\frac{2\pi y}{a}\, d\left(\frac{y}{a}\right) = \frac{1}{4\pi i}\left[\frac{1}{1+2i}\sinh\frac{2\pi y}{a}(1+2i) - \frac{1}{1-2i}\sinh\frac{2\pi y}{a}(1-2i)\right]_0^1$$

$$= \frac{1}{4\pi}\left[-\frac{4}{5}\sinh\frac{2\pi y}{a}\cos\frac{4\pi y}{a} + \frac{2}{5}\cosh\frac{2\pi y}{a}\sin\frac{4\pi y}{a}\right]_0^1 = -\frac{\sinh 2\pi}{5\pi}$$

465

$$\int_0^1 \sin\frac{2\pi y}{a}\sinh\frac{4\pi y}{a}\,d\left(\frac{y}{a}\right) = \frac{1}{4\pi i}\left[\frac{1}{2+i}\sinh\frac{2\pi y}{a}(2+i) - \frac{1}{2-i}\sinh\frac{\pi y}{a}(2-i)\right]_0^1$$

$$= \frac{1}{4\pi}\left[-\frac{2}{5}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} + \frac{4}{5}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right]_0^1 = -\frac{\sinh 4\pi}{10\pi}$$

$$\int_0^1 \frac{4\pi y}{a}\sin\frac{2\pi y}{a}\cosh\frac{4\pi y}{a}\,d\left(\frac{y}{a}\right) = \frac{1}{2\pi i}\left[\frac{1}{(2+i)^2}\left\{\frac{2\pi y}{a}(2+i)\cosh\frac{2\pi y}{a}(2+i) - \sinh\frac{2\pi y}{a}(2+i)\right\}\right.$$

$$\left.-\frac{1}{(2-i)^2}\left\{\frac{2\pi y}{a}(2-i)\cosh\frac{2\pi y}{a}(2-i) - \sinh\frac{2\pi y}{a}(2-i)\right\}\right]_0^1$$

$$= \frac{1}{2\pi}\left[\frac{2\pi y}{a}\left\{-\frac{2}{5}\cosh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} + \frac{4}{5}\sinh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right\} + \left\{\frac{8}{25}\sinh\frac{4\pi y}{a}\cos\frac{2\pi y}{a} - \frac{6}{25}\cosh\frac{4\pi y}{a}\sin\frac{2\pi y}{a}\right\}\right]_0^1$$

$$= -\frac{2}{5}\cosh 4\pi + \frac{4}{25\pi}\sinh 4\pi$$

$$\int_0^1 \left\{(a_2+b_2)\sinh\frac{2\pi y}{a} + b_2\left(\frac{2\pi y}{a}\right)\cosh\frac{2\pi y}{a}\right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{(a_2+b_2)^2}{2\pi}\left\{\frac{\sinh 4\pi}{4} - \pi\right\} + \frac{b_2(a_2+b_2)}{8\pi}\left\{4\pi\cosh 4\pi - \sinh 4\pi\right\} + \frac{b_2^2}{32\pi}\left[\frac{64\pi^3}{3} + (16\pi^2+2)\sinh 4\pi - 8\pi\cosh 4\pi\right]$$

$$\int_0^1 \left\{ (a_4+b_4)\sinh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\cosh\frac{4\pi y}{a} \right\}^2 d\left(\frac{y}{a}\right)$$

$$= \frac{(a_4+b_4)^2}{4\pi}\left\{\frac{\sinh 8\pi}{4} - 2\pi\right\} + \frac{b_4(a_4+b_4)}{16\pi}\left\{8\pi\cosh 8\pi - \sinh 8\pi\right\} + \frac{b_4^2}{64\pi}\left[\frac{(8\pi)^3}{3} + (64\pi^2+2)\sinh 8\pi - 16\pi\cosh 8\pi\right]$$

$$\frac{1}{a^2E^2}\int_0^a\int_0^a \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4\frac{f^2}{256}\left[\frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + 2\left(\frac{f\pi^2}{50}\right)^2\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{64}\left[\frac{1}{4}\left(\frac{f\pi^2}{2}-1\right)\left\{-(a_2+b_2)\frac{\sinh 2\pi}{4\pi} + b_2\left(\frac{\sinh 2\pi}{4\pi} - \frac{1}{2}\cosh 2\pi\right)\right\}\right.$$

$$+\frac{f\pi^2}{50}\left\{-(a_2+b_2)\frac{\sinh 2\pi}{5\pi} + b_2\left(-\frac{2}{5}\cosh 2\pi + \frac{2}{25\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\pi^2}{50}\left\{-(a_4+b_4)\frac{\sinh 4\pi}{10\pi} + b_4\left(-\frac{2}{5}\cosh 4\pi + \frac{4}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{f^2}{128}\left[\frac{(a_2+b_2)^2}{2}\left\{\frac{\sinh 4\pi}{4\pi} - 1\right\} + \frac{b_2(a_2+b_2)}{2}\left\{\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right\} + \frac{b_2^2}{2}\left\{\frac{4\pi^2}{3} + \frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{\cosh 4\pi}{2}\right\}\right.$$

$$\left.+\frac{(a_4+b_4)^2}{2}\left\{\frac{\sinh 8\pi}{8\pi} - 1\right\} + \frac{b_2(a_2+b_2)}{2}\left\{\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right\} + \frac{b_2^2}{2}\left\{\frac{16\pi^2}{3} + \frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{\cosh 8\pi}{2}\right\}\right]$$

$$\frac{1}{2}\left\{\left(\frac{\partial w}{\partial y}\right)^2\right\} = \frac{1}{2}\left(\frac{a}{R}\right)^4\left[\frac{f}{4}\left(1+\cos\frac{2\pi x}{a}\right)\left(\frac{\pi}{a}\right)\sin\frac{2\pi y}{a}\right]^2 R^2$$

$$= \frac{1}{2}\left(\frac{a}{R}\right)^2\frac{f}{4}\left[\frac{f\pi^2}{4}\left(1+\cos\frac{2\pi x}{a}\right)^2\sin^2\frac{2\pi y}{a}\right] = \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\frac{f\pi^2}{4}\left(\frac{3}{2}+2\cos\frac{2\pi x}{a}+\frac{1}{2}\cos\frac{4\pi x}{a}\right)\left(\frac{1}{2}-\frac{1}{2}\cos\frac{4\pi y}{a}\right)\right]$$

$$\frac{\partial v}{\partial y} = \frac{\sigma_y}{E} - \frac{1}{2}\left\{\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$= -\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2\frac{f}{8}\left[-\frac{3}{16}f\pi^2 + \frac{3}{16}f\pi^2\cos\frac{4\pi y}{a}\right) + \left[\left\{\frac{f}{4}\left(\frac{f\pi^2}{2}-1\right)\cos\frac{2\pi y}{a} + \frac{26}{100}f\pi^2\cos\frac{4\pi y}{a}\right\} - \left\{a_2\cosh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\sinh\left(\frac{2\pi y}{a}\right)\right\}\right]\cos\frac{2\pi x}{a}$$

$$+ \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\left\{\frac{f\pi^2}{25}\cos\frac{2\pi y}{a} + \frac{f\pi^2}{16}\cos\frac{4\pi y}{a}\right\} - \left\{a_4\cosh\frac{4\pi y}{a} + b_4\left(\frac{4\pi y}{a}\right)\sinh\frac{4\pi y}{a}\right\}\right]\cos\frac{4\pi x}{a}$$

$$\frac{v}{a} = -\frac{\sigma}{E}\left(\frac{y}{a}\right) + \left(\frac{a}{R}\right)^2\frac{f}{8}\left[-\frac{3}{16}f\pi^2\left(\frac{y}{a}\right) + \frac{3}{64\pi}f\pi^2\sin\frac{4\pi y}{a} + \right.$$

$$\left\{\frac{1}{8\pi}\left(\frac{f\pi^2}{2}-1\right)\sin\frac{2\pi y}{a} + \frac{26}{400\pi}f\pi^2\sin\frac{4\pi y}{a}\right\}\cos\frac{2\pi x}{a} - \frac{1}{2\pi}\left\{(a_2-b_2)\sinh\left(\frac{2\pi y}{a}\right) + b_2\left(\frac{2\pi y}{a}\right)\cosh\left(\frac{2\pi y}{a}\right)\right\}\cos\frac{2\pi x}{a}$$

$$+ \left(\frac{a}{R}\right)^2\frac{f}{8}\left[\left\{\frac{f\pi^2}{50\pi}\sin\frac{2\pi y}{a} + \frac{f\pi^2}{64\pi}\sin\frac{4\pi y}{a}\right\} - \frac{1}{4\pi}\left\{(a_4-b_4)\sinh\left(\frac{4\pi y}{a}\right) + b_4\left(\frac{4\pi y}{a}\right)\cosh\left(\frac{4\pi y}{a}\right)\right\}\right]\cos\frac{4\pi x}{a}$$

468

At $y = \pm a$,

$$\left(\frac{v}{a}\right)_a = -\frac{\sigma}{E} + \left(\frac{a}{R}\right)^2 \frac{f}{8}\left[-\frac{3}{16} f\pi^2 - \frac{1}{2\pi}\left\{(a_2-b_2)\sinh 2\pi + 2\pi b_2 \cosh 2\pi\right\}\cos\frac{2\pi x}{a} - \frac{1}{4\pi}\left\{(a_4-b_4)\sinh 4\pi + 4\pi b_4 \cosh 4\pi\right\}\cos\frac{4\pi x}{a}\right]$$

The potential increase $\Delta\wp$ is

$$a t\int_0^a \sigma\left(\frac{v}{a}\right)_a dx \quad , \quad \text{or} \quad \boxed{\frac{\Delta\wp}{a^2 t E} = -\left(\frac{a}{R}\right)^2 \frac{f}{8}\frac{\sigma}{E}\left(\frac{3}{16} f\pi^2\right)} \qquad \left[-\left(\frac{\sigma}{E}\right)^2\right.$$

cancel with term in σ_y energy

$$\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2} = \frac{1}{R}\left[\frac{f}{2}\pi^2 \cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\right]$$

$$\frac{t^2}{12a^2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial^2 w_0}{\partial x^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\int_0^a\int_0^a\left\{\cos\frac{2\pi x}{a}\left(1+\cos\frac{2\pi y}{a}\right)\right\}^2 d\left(\frac{x}{a}\right)d\left(\frac{y}{a}\right)$$

$$= \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2 \frac{1}{4}(2+1) = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12a^2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial y^2}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{3}{4}$$

$$\frac{t^2}{12a^2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2 dx\,dy = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{2}\right)^2\frac{1}{4}$$

$$\cosh 2\pi \cdot a_2 + 2\pi \sinh 2\pi \cdot b_2 = \frac{77}{200}(f\pi^2) - \frac{1}{4}$$

$$\sinh 2\pi \cdot a_2 + (\sinh 2\pi + 2\pi \cosh 2\pi)\, b_2 = 0$$

$$2(\sinh 4\pi + 4\pi)\, b_2 = \sinh 2\pi \left(1 - \frac{77}{50} f\pi^2\right)$$

$$\boxed{b_2 = \frac{\sinh 2\pi}{2(\sinh 4\pi + 4\pi)}\left(1 - \frac{77}{50} f\pi^2\right)}$$

$$a_2 = -\, b_2 \,\frac{\sinh 2\pi + 2\pi \cosh 2\pi}{\sinh 2\pi}$$

$$\boxed{a_2 = -\frac{\sinh 2\pi + 2\pi \cosh 2\pi}{2(\sinh 4\pi + 4\pi)}\left(1 - \frac{77}{50} f\pi^2\right)}$$

$$\cosh 4\pi \; a_4 + 4\pi \sinh 4\pi \cdot b_4 = \frac{41}{400}(f\pi^2)$$

$$\sinh 4\pi \; a_4 + (\sinh 4\pi + 4\pi \cosh 4\pi)\, b_4 = 0$$

$$\frac{1}{2}(\sinh 8\pi + 8\pi)\, b_4 = -\frac{41}{400}(f\pi^2) \sinh 4\pi$$

$$\boxed{b_4 = -\frac{\sinh 4\pi}{(\sinh 8\pi + 8\pi)} \frac{41}{200}(f\pi^2)}$$

$$\boxed{a_4 = +\frac{\sinh 4\pi + 4\pi \cosh 4\pi}{(\sinh 8\pi + 8\pi)} \frac{41}{200}(f\pi^2)}$$

The extensional energy $= \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2\right]$

$$2\,\frac{\text{The extensional energy}}{Ea^2t} = \left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[\frac{1}{32}(f\pi^2)^2 + \frac{1}{2}\left(\frac{f\pi^2}{4}-1\right)^2 + \frac{1}{16}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f\pi^2)^2}{20000} + \frac{1}{256}(f\pi^2)^2 + \frac{3}{2500}(f\pi^2)^2\right]$$

$$+\left(\frac{a}{R}\right)^4 \frac{f^2}{64}\left[-a_2\,\frac{3}{4}\,\frac{\sinh 2\pi}{2\pi}\left\{1+\frac{9}{50}(f\pi^2)\right\} - b_2\left\{\left(\frac{f}{4}\,\frac{\sinh 2\pi}{2\pi} + \frac{3}{4}\cosh 2\pi\right) + \left(\frac{139}{1000}\cosh 2\pi - \frac{3123}{5000}\,\frac{\sinh 2\pi}{2\pi}\right)(f\pi^2)\right\}\right.$$

$$\left. - a_4\,\frac{f\pi^2}{25}\,\frac{\sinh 4\pi}{4\pi} - b_4\left\{\cdot\cosh 4\pi - \frac{33}{8}\,\frac{\sinh 4\pi}{4\pi}\right\}\frac{(f\pi^2)}{25}\right]$$

$$+\left(\frac{a}{R}\right)^2 \frac{f^2}{64}\left[\frac{\sinh 4\pi}{4\pi}\,a_2^2 + \left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi\right) a_2 b_2 + \frac{1}{2}\left\{(8\pi^2+2)\,\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi + 1\right\} b_2^2\right.$$

$$\left. + \frac{\sinh 8\pi}{8\pi}\,a_4^2 + \left(\frac{\sinh 8\pi}{8\pi} + \cosh 8\pi\right) a_4 b_4 + \frac{1}{2}\left\{(32\pi^2+2)\,\frac{\sinh 8\pi}{8\pi} + \cosh 8\pi + 1\right\} b_4^2\right]$$

$$2\,\frac{\Delta p}{Ea^2t} = -\left(\frac{a}{R}\right)^2 \frac{f^2}{64}\left[3\pi^2\,\frac{\sigma}{E}\right]$$

$$2\,\frac{\text{The bending energy}}{Ea^2t} = \frac{1}{6}\left(\frac{t}{R}\right)^2 \frac{(f\pi^2)^2}{4}$$

47

a_2^2	$a_2 b_2$	b_2^2
$\frac{\sinh 4\pi}{4\pi} + 1$ λ^4	$4\,\frac{\sinh 4\pi}{4\pi} + 4$ $\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}$ λ^4	$4\,\frac{\sinh 4\pi}{4\pi} + 4$ $2\cosh 4\pi - 2\,\frac{\sinh 4\pi}{4\pi}$ λ^0 $\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}$
$\frac{\sinh 4\pi}{4\pi} + 1$	$\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}$	$\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^4}{3}$
$2\,\frac{\sinh 4\pi}{4\pi} - 2$ λ^2	$4\,\frac{\sinh 4\pi}{4\pi} - 4$ $2\cosh 4\pi - 2\,\frac{\sinh 4\pi}{4\pi}$ λ^2	$2\,\frac{\sinh 4\pi}{4\pi} - 2$ $2\cosh 4\pi - 2\,\frac{\sinh 4\pi}{4\pi}$ $2\,\frac{4\pi^2}{3} - \cosh 4\pi + 2\,\frac{16\pi^2+2}{16\pi}\sinh 4\pi$ λ^2
$4\,\frac{\sinh 4\pi}{4\pi}$	$4\,\frac{\sinh 4\pi}{4\pi} + 4\cosh 4\pi +$	$(16\pi^2+4)\,\frac{\sinh 4\pi}{4\pi} + 2\cosh 4\pi + 2$

$2\frac{1}{2}\lambda^4 + \frac{1}{2} + \lambda^2$

$$(f\pi^2)^2\left[-\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{20{,}000}+\frac{1}{256}+\frac{3}{2500}\right]=(f\pi^2)^2\left[0.08203125+0.00125000\right]$$

$$+(f\pi^2)\left[-\frac{1}{4}-\frac{1}{16}\right]=-(f\pi^2)\left[0.3125000\right]$$

$$+\left[\frac{1}{2}+\frac{1}{16}\right]=0.5625000$$

$$2W_1=\left(\frac{a}{R}\right)^4\frac{f^2}{64}\Big[0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$$

$$-0.75\,\frac{\sinh 2\pi}{2\pi}\left\{1+0.18000(f\pi^2)\right\}a_2-\left\{\left(1.2500\,\frac{\sinh 2\pi}{2\pi}+0.7500\cosh 2\pi\right)+\left(0.13900\cosh 2\pi-0.624600\,\frac{\sinh 2\pi}{2\pi}\right)(f\pi^2)\right\}b_2$$

$$-0.0400\,\frac{\sinh 4\pi}{4\pi}(f\pi^2)\,a_4-\left\{0.04000\cosh 4\pi-0.165000\,\frac{\sinh 4\pi}{4\pi}\right\}(f\pi^2)\,b_4$$

$$+\frac{\sinh 4\pi}{4\pi}a_2^2+\left(\frac{\sinh 4\pi}{4\pi}+\cosh 4\pi\right)a_2b_2+\left\{(4\pi^2+1)\frac{\sinh 4\pi}{4\pi}+0.5\cosh 4\pi+0.5\right\}b_2^2+$$

$$+\frac{\sinh 8\pi}{8\pi}a_4^2+\left(\frac{\sinh 8\pi}{8\pi}+\cosh 8\pi\right)a_4b_4+\left\{(16\pi^2+1)\frac{\sinh 8\pi}{8\pi}+0.5\cosh 8\pi+0.5\right\}b_4^2\Big]$$

$$2W_2=-\left(\frac{a}{R}\right)^2\frac{f^2}{64}\left[3\pi^2\frac{\sigma}{E}\right]$$

$$2W_3=\left(\frac{t}{R}\right)^2\frac{f^2}{64}\left[\frac{16\pi^4}{6}\right]$$

473

$$a_2 = -\frac{\left(\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi\right)}{\left(\frac{\sinh 4\pi}{4\pi} + 1\right)}\left(0.25000 - 0.385000 f\pi^2\right)$$

$$b_2 = +\frac{\frac{\sinh 2\pi}{2\pi}}{\left(\frac{\sinh 4\pi}{4\pi} + 1\right)}\left(0.25000 - 0.385000 f\pi^2\right)$$

$$a_4 = +\frac{\left(\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi\right)}{\left(\frac{\sinh 8\pi}{8\pi} + 1\right)}\, 0.1025000 (f\pi^2)$$

$$b_4 = -\frac{\left(\frac{\sinh 4\pi}{4\pi}\right)}{\left(\frac{\sinh 8\pi}{8\pi} + 1\right)}\, 0.102500 (f\pi^2)$$

$$\log_e (e^{\xi}) = \xi = 2.30258509 \log_{10} (e^{\xi})$$

$$\log_{10} (e^{\xi}) = 0.434294482\ \xi$$

$$\therefore \log_{10} (e^{2\pi}) = 0.434294482 \times 6.283185307 = 2.72\text{[illegible]}$$

$$e^{2\pi} = 535.49162, \qquad e^{-2\pi} = 0.00187$$

$$\sinh 2\pi = 267.74675 \qquad \cosh 2\pi = 267.74862$$

$$\boxed{\frac{\sinh 2\pi}{2\pi} = 42.613218 \qquad \cosh 2\pi = 267.74862}$$

$$\log_{10}(e^{4\pi}) = 5.45750542,$$

$$e^{4\pi} = 286751.33 \qquad \sinh 4\pi = 143375.66$$

$$\boxed{\frac{\sinh 4\pi}{4\pi} = 11409.473, \qquad \cosh 4\pi = 143375.66}$$

$$\log_{10}(e^{8\pi}) = 10.91501083, \qquad e^{8\pi} = 82226314000$$

$$\boxed{\frac{\sinh 8\pi}{8\pi} = 1635840500 \qquad \cosh 8\pi = 41113157000}$$

$$a_2 = -\,0.027199735\,(0.2500 - 0.38500\,f\pi^2)$$

$$b_2 = +\,0.0037345707\,(0.2500 - 0.38500\,f\pi^2)$$

$$a_4 = +\,0.000094621164\,(0.1025000 - f\pi^2)$$

$$b_4 = -\,0.00000697468\ldots\,(0.1025000 - f\pi^2)$$

$$+0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$$

$$+(0.25000-0.385000f\pi^2)\{0.86930118(1+0.1800f\pi^2)-(0.94887221+0.03958959f\pi^2)\}$$

$$-0.1025000(f\pi^2)^2\{0.043183105-0.026869720\}+$$

$$+(0.2500-0.38500f\pi^2)^2\{8.4410200-15.7229706+7.4410933\}$$

$$+(0.1025000f\pi^2)^2\{14.6459494-28.2123232+13.6459506\}$$

$$=0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$$

$$-(0.25000-0.385000f\pi^2)(0.07957103-0.11688462f\pi^2)-0.001672122(f\pi^2)^2$$

$$+0.1591427(0.2500-0.38500f\pi^2)^2+0.0008360536(f\pi^2)^2$$

$$=0.08328125(f\pi^2)^2-0.3125000(f\pi^2)+0.5625000$$

$$-(0.0198928-0.0598560f\pi^2+0.0450006f^2\pi^4)-0.001672122(f\pi^2)^2$$

$$+(0.00994642-0.03063497f\pi^2+0.02358893f^2\pi^4)+0.0008360536(f\pi^2)^2$$

$$=0.06103351(f\pi^2)^2-0.28327897(f\pi^2)+0.5525362$$

$$6\pi^2 \frac{\sigma}{E}\left(\frac{a}{R}\right)^2 = \left(\frac{a}{R}\right)^4\left[0.24413404\,\pi^4 f^2 - 0.84983691\,\pi^2 f + 1.1050724\right] \qquad \underline{\underline{477}}$$

$$+ \left(\frac{t}{R}\right)^2 \frac{16\pi^4}{3}$$

$$K = \gamma^2\left[0.040689\,\pi^2 f^2 - 0.1416395 f + \frac{0.1841787}{\pi^2}\right] + \frac{8}{9}\pi^2 \frac{1}{\gamma^2}$$

now

$$\frac{\delta}{t} = \frac{R}{t}\frac{1}{2}\left(\frac{a}{R}\right)^2 f, \quad \text{or} \quad f = \frac{2\left(\frac{\delta}{t}\right)}{\gamma^2}$$

$$\therefore \quad K = \pi^2\left[0.162756\left(\frac{\delta}{t}\right)^2 + \frac{8}{9}\right]\frac{1}{\gamma^2} - 0.2832790\left(\frac{\delta}{t}\right) + \frac{0.1841787}{\pi^2}\gamma^2$$

$$\text{Thus} \quad K = 2\left[0.0299762\left(\frac{\delta}{t}\right)^2 + 0.1637144\right]^{\frac{1}{2}} - 0.2832790\left(\frac{\delta}{t}\right)$$

$$\frac{0.0599524\left(\frac{\delta}{t}\right)}{\left[0.0299762\left(\frac{\delta}{t}\right)^2 + 0.1637144\right]^{\frac{1}{2}}} = 0.2832790$$

$$0.00359429\left(\frac{\delta}{t}\right)^2 = 0.00240550\left(\frac{\delta}{t}\right)^2 + 0.01313759$$

$$\left(\frac{\delta}{t}\right)^2_{min} = \frac{0.01313759}{0.00118879} = 11.05123, \qquad \left(\frac{\delta}{t}\right)_{min} = 3.32434$$

$$K_{min} = \left(\frac{0.1199048}{0.2832790} - 0.2832790\right) 3.32434 = 0.4653930$$

40
1600

$$\gamma = \pi \left[0.883685 \left(\frac{\delta}{t}\right)^2 + 4.826230 \right]^{\frac{1}{4}}$$

$$\gamma_0 = \pi \sqrt[4]{4.826230} = \pi \sqrt[2]{2.196868} = 1.482183\pi$$

$$\gamma_{min} = \pi \sqrt[4]{14.592036} = \pi \sqrt[2]{3.819952} = 1.954469\pi$$

①	②	③	④	⑤	⑥		
$\left(\frac{\delta}{t}\right)$	$0.2832790\left(\frac{\delta}{t}\right)$	$0.0299762\left(\frac{\delta}{t}\right)^2$	③ $+ 0.1637144$	④$^{\frac{1}{2}}$	K		
0	0	0	0.1637144	0.404617	0.809234		
1	0.2832790	0.0299762	0.1936906	0.440104	0.596929		
2	0.5665580	0.1199048	0.2836191	0.532559	0.498560		
3	0.8498370	0.2697858	0.4335002	0.658407	0.466977		
4	1.1331160	0.4796192	0.6433336	0.802081	0.471046		
5	1.4163950	0.7494050	0.9131194	0.955573	0.494751		
6	1.6996740	1.0791432	1.2421526	1.1148352	0.529996		
7	1.9829530	1.4688338	1.6325482	1.277713	0.572473		
8	2.26[illegible]320	1.91[illegible]	2.0[illegible]	1.442979	0.619726		
9	2.5495110	2.4280722					
10	2.8327900	2.9976200					
11	3.1116690	3.6271202					

$\lambda = 1.0000$

If the wave length in y- direction is (a/λ) instead of a

$$\nabla^4 F = \frac{E}{R^2}\frac{(f\lambda^2\pi^2)}{2}\left[-\frac{f\pi^2}{4}\cos\frac{2\pi x}{a} + \left\{1-\frac{f\pi^2}{4}\right\}\cos\frac{2\pi\lambda y}{a} + \left\{1-\frac{f\pi^2}{2}\right\}\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left. - \frac{f\pi^2}{4}\left\{\cos\frac{4\pi x}{a} + \cos\frac{4\pi\lambda y}{a} + \cos\frac{4\pi x}{a}\cos\frac{2\pi\lambda y}{a} + \cos\frac{2\pi x}{a}\cos\frac{4\pi\lambda y}{a}\right\}\right]$$

Therefore the particular integral is

$$E\left(\frac{a}{R}\right)^2\frac{f\lambda^2}{8}\left[-\frac{f\pi^2}{4}\frac{\cos\frac{2\pi x}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{\lambda^4}\left\{1-\frac{f\pi^2}{4}\right\}\frac{\cos\frac{2\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{(1+\lambda^2)^2}\left\{1-\frac{f\pi^2}{2}\right\}\frac{\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2}\right.$$

$$\left. -\frac{f\pi^2}{4}\left\{\frac{1}{4}\frac{\cos\frac{4\pi x}{a}}{\left(\frac{4\pi}{a}\right)^2} + \frac{1}{4\lambda^4}\frac{\cos\frac{4\pi\lambda y}{a}}{\left(\frac{4\pi}{a}\right)^2} + \frac{1}{(4+\lambda^2)^2}\frac{\cos\frac{4\pi x}{a}\cos\frac{2\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2} + \frac{1}{(1+4\lambda^2)^2}\frac{\cos\frac{2\pi x}{a}\cos\frac{4\pi\lambda y}{a}}{\left(\frac{2\pi}{a}\right)^2}\right\}\right]$$

$$F = \frac{E}{\left(\frac{2\pi}{a}\right)^2}\left[A_2\cosh\left(\frac{2\pi\lambda y}{a}\right) + B_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\left(\frac{2\pi\lambda y}{a}\right)\right]\cos\frac{2\pi x}{a}$$

$$+ \frac{E}{\left(\frac{4\pi}{a}\right)^2}\left[A_4\cosh\left(\frac{4\pi\lambda y}{a}\right) + B_2\left(\frac{4\pi\lambda y}{a}\right)\sinh\left(\frac{4\pi\lambda y}{a}\right)\right]\cos\frac{4\pi x}{a} + Ea^2\left(\frac{x}{a}\right)^2 C$$

419

$$\frac{\sigma_y}{E} = -\left[A_2 \cosh\left(\frac{2\pi\lambda y}{a}\right) + B_2\left(\frac{2\pi\lambda y}{a}\right)\sinh\frac{2\pi\lambda y}{a}\right]\cos\frac{2\pi x}{a} - \left[A_4 \cosh\left(\frac{4\pi\lambda y}{a}\right) + B_4\left(\frac{4\pi\lambda y}{a}\right)\sinh\left(\frac{4\pi\lambda y}{a}\right)\right]\cos\frac{4\pi x}{a}$$

$$+ 2C + \left(\frac{a}{R}\right)^2 \frac{f\lambda^2}{8}\left[-\frac{f\pi^2}{4}\cos\frac{2\pi x}{a} - \frac{1}{(1+\lambda^2)^2}\left(1-\frac{f\pi^2}{2}\right)\cos\frac{2\pi x}{a}\cos\frac{2\pi\lambda y}{a}\right.$$

$$\left. + \frac{f\pi^2}{4}\left\{\frac{1}{4}\cos\frac{4\pi x}{a} + \frac{4}{(4+\lambda^2)^2}\cos\frac{4\pi x}{a}\cos\frac{2\pi\lambda y}{a} + \frac{1}{(1+4\lambda^2)^2}\cos\frac{2\pi x}{a}\cos\frac{4\pi\lambda y}{a}\right\}\right]$$

$a_2 \cosh 2\pi + 2\pi b_2 \sinh 2\pi = \frac{f\pi^2}{4} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{2}-1\right) + \frac{1}{4(1+4\lambda^2)^2} f\pi^2$
$(a_2 + b_2)\sinh 2\pi + 2\pi b_2 \cosh 2\pi = 0$
$a_4 \cosh 4\pi + 4\pi b_4 \sinh 4\pi = \frac{f\pi^2}{4}\left(\frac{1}{4} + \frac{4}{(4+\lambda^2)^2}\right)$
$(a_4 + b_4)\sinh 4\pi + 4\pi b_4 \cosh 4\pi = 0$

$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^4 \frac{(f\lambda^2)^2}{256}\left[\frac{1}{8}(f\pi^2)^2 + \frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f\pi^2)^2}{(4+\lambda^2)^4} + \frac{(f\pi^2)^2}{16(1+4\lambda^2)^4} + \frac{(f\pi^2)^2}{128}\right]$$

$$-\left(\frac{a}{R}\right)^4\frac{(f\lambda^2)^2}{64}\left[\frac{f\pi^2}{4}\left\{\frac{(a_2-b_2)}{2\pi}\sinh 2\pi + b_2\cosh 2\pi\right\} + \frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{2}-1\right)\left\{a_2\frac{\sinh 2\pi}{4\pi} + b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+\frac{f\pi^2}{4(1+4\lambda^2)^2}\left\{\frac{1}{10\pi}a_2\sinh 2\pi + b_2\left(\frac{1}{5}\cosh 2\pi + \frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\pi^2}{16}\left\{\frac{(a_4-b_4)}{4\pi}\sinh 4\pi + b_4\cosh 4\pi\right\} + \frac{f\pi^2}{(1+\lambda^2)^2}\left\{a_4\frac{\sinh 4\pi}{5\pi} + b_4\left(\frac{4}{5}\cosh 4\pi - \frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^4\frac{(f\lambda^2)^2}{128}\left[\frac{a_2^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right) + \frac{a_2 b_2}{2}\left(\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right) + \frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{1}{2}\cosh 4\pi - \frac{4\pi^2}{3}\right)\right.$$

$$\left.+\frac{a_4^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right) + \frac{a_4 b_4}{2}\left(\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right) + \frac{b_4^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{1}{2}\cosh 8\pi - \frac{16\pi^2}{3}\right)\right]$$

$$= A_1$$

44

$$\frac{1}{abE^2}\int_0^a\int_0^b \sigma_x^2\,dx\,dy = \lambda^4\left(\frac{a}{R}\right)^4\frac{(f\lambda^2)^2}{256}\left[\frac{2}{\lambda^8}\left(\frac{f\pi^2}{4}-1\right)^2+\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{2}-1\right)^2+\frac{2}{\lambda^8}\left(\frac{f\pi^2}{16}\right)^2+\frac{(f\pi^2)^2}{16(4+\lambda^2)^4}\right.$$

$$\left.+\frac{(f\pi^2)^2}{(1+4\lambda^2)^4}\right]$$

$$+\lambda^4\left(\frac{a}{R}\right)^4\frac{(f\lambda^2)^2}{64}\left[\frac{1}{\lambda^4}\left(\frac{f\pi^2}{4}-1\right)\left\{\frac{(a_2+b_2)}{2\pi}\sinh 2\pi+b_2\cosh 2\pi\right\}+\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{2}-1\right)\left\{(a_2+2b_2)\frac{\sinh 2\pi}{4\pi}+b_2\frac{\cosh 2\pi}{2}\right\}\right.$$

$$+\frac{f\pi^2}{(1+4\lambda^2)^2}\left\{\frac{1}{10\pi}(a_2+2b_2)\sinh 2\pi+b_2\left(\frac{1}{5}\cosh 2\pi+\frac{3}{50\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\pi^2}{16\lambda^4}\left\{\frac{(a_4+b_4)}{4\pi}\sinh 4\pi+b_4\cosh 4\pi\right\}+\frac{f\pi^2}{4(4+\lambda^2)^2}\left\{\frac{1}{5\pi}(a_4+2b_4)\sinh 4\pi+b_4\left(\frac{4}{5}\cosh 4\pi-\frac{3}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\left(\frac{a}{R}\right)^4\lambda^4\frac{(f\lambda)^2}{128}\left[\frac{(a_2+2b_2)^2}{2}\left(\frac{\sinh 4\pi}{4\pi}+1\right)+\frac{(a_2+2b_2)b_2}{2}\left(\cosh 4\pi-\frac{\sinh 4\pi}{4\pi}\right)+\frac{b_2^2}{2}\left(\frac{16\pi^2+2}{16\pi}\sinh 4\pi-\frac{1}{2}\cosh 4\pi-\frac{4\pi^2}{3}\right)\right.$$

$$\left.+\frac{(a_4+2b_4)^2}{2}\left(\frac{\sinh 8\pi}{8\pi}+1\right)+\frac{(a_4+2b_4)b_4}{2}\left(\cosh 8\pi-\frac{\sinh 8\pi}{8\pi}\right)+\frac{b_4^2}{2}\left(\frac{64\pi^2+2}{32\pi}\sinh 8\pi-\frac{1}{2}\cosh 8\pi-\frac{16\pi^2}{3}\right)\right]$$

$$= A_2$$

42

$$\frac{1}{abE^2}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \lambda^2\left(\frac{a}{R}\right)^4 \frac{(f\lambda)^2}{256}\left[\frac{1}{(1+\lambda^2)^4}\left(\frac{f\pi^2}{2}-1\right)^2 + \frac{(f\pi^2)^2}{4(4+\lambda^2)^4} + \frac{(f\pi^2)^2}{4(1+4\lambda^2)^4}\right]$$

$$+\lambda^2\left(\frac{a}{R}\right)^4 \frac{(f\lambda)^2}{64}\left[\frac{1}{(1+\lambda^2)^2}\left(\frac{f\pi^2}{2}-1\right)\left\{-(a_2+b_2)\frac{\sinh 2\pi}{4\pi} + b_2\left(\frac{\sinh 2\pi}{4\pi} - \frac{1}{2}\cosh 2\pi\right)\right\}\right.$$

$$+\frac{f\pi^2}{2(1+4\lambda^2)^2}\left\{-(a_2+b_2)\frac{\sinh 2\pi}{5\pi} + b_2\left(-\frac{2}{5}\cosh 2\pi + \frac{2}{25\pi}\sinh 2\pi\right)\right\}$$

$$\left.+\frac{f\lambda\pi^2}{2(4+\lambda^2)^2}\left\{-(a_4+b_4)\frac{\sinh 4\pi}{10\pi} + b_4\left(-\frac{2}{5}\cosh 4\pi + \frac{4}{25\pi}\sinh 4\pi\right)\right\}\right]$$

$$+\lambda^2\left(\frac{a}{R}\right)^4 \frac{(f\lambda)^2}{128}\left[\frac{(a_2+b_2)^2}{2}\left\{\frac{\sinh 4\pi}{4\pi}-1\right\} + \frac{b_2(a_2+b_2)}{2}\left\{\cosh 4\pi - \frac{\sinh 4\pi}{4\pi}\right\} + \frac{b_2^2}{2}\left\{\frac{4\pi^2}{3} + \frac{16\pi^2+2}{16\pi}\sinh 4\pi - \frac{\cosh 4\pi}{2}\right\}\right.$$

$$\left.+\frac{(a_4+b_4)^2}{2}\left\{\frac{\sinh 8\pi}{8\pi}-1\right\} + \frac{b_4(a_4+b_4)}{2}\left\{\cosh 8\pi - \frac{\sinh 8\pi}{8\pi}\right\} + \frac{b_4^2}{2}\left\{\frac{16\pi^2}{3} + \frac{64\pi^2+2}{32\pi}\sinh 8\pi - \frac{\cosh 8\pi}{2}\right\}\right]$$

A_3

483

$$\frac{1}{a}\int_0^a \sin\frac{2\pi x}{a}\sinh(\sqrt{2}\lambda)\frac{\pi x}{a}\,dx = \frac{1}{2\pi i}\int_0^a \left[\cosh\frac{\pi x}{a}(\sqrt{2}\lambda+2i) - \cosh\frac{\pi x}{a}(\sqrt{2}\lambda-2i)\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi i}\left[\frac{1}{\sqrt{2}\lambda+2i}\sinh\pi(\sqrt{2}\lambda+2i) - \frac{1}{\sqrt{2}\lambda-2i}\sinh\pi(\sqrt{2}\lambda-2i)\right] = -\frac{2}{\pi}\,\frac{\sinh\pi\sqrt{2}\lambda}{4+2\lambda^2} = -\frac{1}{\pi}\,\frac{\sinh\pi\sqrt{2}\lambda}{2+\lambda^2}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sin\frac{2\pi x}{a}\,dx = \frac{1}{4\pi}\Big[\sin\theta - \theta\cos\theta\Big]_0^{2\pi} = -\frac{1}{2}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sin\frac{2\pi x}{a}\cos\frac{\pi x}{a}\,dx = \frac{1}{2\pi}\int_0^a \frac{\pi x}{a}\left[\sin\frac{3\pi x}{a} + \sin\frac{\pi x}{a}\right] d\left(\frac{\pi x}{a}\right)$$

$$= \frac{1}{2\pi}\left[\frac{1}{9}\left(\sin\theta - \theta\cos\theta\right)_0^{3\pi} + \left(\sin\theta - \theta\sin\theta\right)_0^{\pi}\right] = \frac{1}{2}\left[\frac{1}{3}+1\right] = \frac{2}{3}$$

$$\frac{1}{a}\int_0^a \sinh^2\frac{\sqrt{2}\lambda\pi x}{a}\,dx = \frac{1}{2\pi}\int_0^{\pi}\left(\cosh 2\sqrt{2}\lambda\theta - 1\right)d\theta = \frac{1}{2\pi}\left[\frac{\sinh 2\sqrt{2}\lambda\pi}{2\sqrt{2}\lambda} - \pi\right] = \frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi} - \frac{1}{2}$$

$$\frac{1}{a}\int_0^a \frac{\pi x}{a}\sinh(\sqrt{2}\lambda)\frac{\pi x}{a}\,dx = \frac{1}{2\lambda^2\pi}\left[\sqrt{2}\lambda\pi\cosh\sqrt{2}\lambda\pi - \sinh\sqrt{2}\lambda\pi\right]$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sinh\sqrt{2}\lambda\left(\frac{\pi x}{a}\right)dx = \frac{1}{2\pi}\int_0^{\pi}\theta\left[\sinh\theta(\sqrt{2}\lambda+i) + \sinh\theta(\sqrt{2}\lambda-i)\right]d\theta$$

$$= \frac{1}{2\pi}\left[\frac{1}{(\sqrt{2}\lambda+i)^2}\left\{(\sqrt{2}\lambda+i)\pi\cosh\pi(\sqrt{2}\lambda+i) - \sinh\pi(\sqrt{2}\lambda+i)\right\} + \frac{1}{(\sqrt{2}\lambda-i)^2}\left\{(\sqrt{2}\lambda-i)\pi\cosh\pi(\sqrt{2}\lambda-i) - \sinh\pi(\sqrt{2}\lambda-i)\right\}\right]$$

$$= -\frac{\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2} + \frac{1}{\pi}\,\frac{(2\lambda^2-1)\sinh\sqrt{2}\lambda\pi}{(1+2\lambda^2)^2}$$

484

$$\frac{1}{a}\int_0 \left(\frac{\pi x}{a}\right)^4 dx = \pi^2 \int_0^1 \theta^2 d\theta = \frac{\pi^2}{3}$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right)^2 \cos\frac{\pi x}{a}\, dx = \frac{1}{\pi}\int_0^\pi \theta^2 \cos\theta\, d\theta = \frac{1}{\pi}\left[-2\pi\right] = -2$$

$$\frac{1}{a}\int_0^a \left(\frac{\pi x}{a}\right)^2 \cos^2\frac{\pi x}{a}\, dx = \frac{1}{\pi}\int_0^\pi \theta^2 \cos^2\theta\, d\theta = \frac{1}{2\pi}\int_0^\pi \theta^2 (1+\cos 2\theta)\, d\theta = \frac{1}{2\pi}\left[\frac{\pi^3}{3} + \frac{1}{8}(4\pi)\right] = \frac{\pi^2}{6} + \frac{1}{4}$$

$$\frac{1}{E^2 ab}\int_0^a\int_0^b \tau_{xy}^2\, dx\, dy = \left(\frac{a}{R}\right)^4 \left(\frac{f\lambda}{8}\right)^2 \Bigg[\frac{1}{4}\left\{ \frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{(1+2\lambda^2)^2} \right\}^2 + \frac{1}{4}\left(\frac{f\pi^2}{32}\right)^2 \left(\frac{\lambda^2}{2+\lambda^2}\right)^2 + \frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2 \left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2$$

$$+ \frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2 \left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2 + \frac{1}{4}\left(\frac{f\pi^2}{16}\right)^2 \left(\frac{3\lambda^2}{1+8\lambda^2}\right)^2 + \frac{1}{4}\left(\frac{f\pi^2}{128}\right)^2 \left(\frac{3\lambda^2}{1+2\lambda^2}\right)^2$$

$$+ \left[\frac{f\pi^2}{8}\frac{\lambda^2}{(1+2\lambda^2)} + \frac{1+\lambda^2}{(1+2\lambda^2)^2}\right]\left\{\frac{1}{2}\frac{1}{(1+2\lambda^2)^2} - \frac{1}{2} + \frac{1}{4}\frac{\lambda^2}{(1+2\lambda^2)}\right\}$$

$$+ \frac{f\pi^2}{32}\frac{\lambda^2}{(2+\lambda^2)}\left\{-\frac{1}{2}\frac{1}{(1+2\lambda^2)(2+\lambda^2)} + \frac{1}{4} - \frac{2}{3}\frac{\lambda^2}{(1+2\lambda^2)}\right\}$$

$$+ \frac{1}{2}\left\{ \frac{\left(\frac{\sinh 2\sqrt{2}\pi}{4\sqrt{2}\lambda\pi} - \frac{1}{2}\right)\pi^2}{4(1+2\lambda)^2 \sinh^2(\sqrt{2}\lambda\pi)} - \frac{1}{2}\frac{\pi}{(1+2\lambda^2)\sinh(\sqrt{2}\lambda\pi)}\frac{1}{2\lambda^2\pi}\left[\sqrt{2}\lambda\pi\cosh\sqrt{2}\lambda\pi - \sinh\sqrt{2}\lambda\pi\right]\right.$$

$$- \frac{\lambda^2}{(1+2\lambda^2)}\frac{\pi}{(1+2\lambda^2)\sinh(\sqrt{2}\lambda\pi)}\left[-\frac{\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2} + \frac{1}{\pi}\frac{(2\lambda^2-1)\sinh\sqrt{2}\lambda\pi}{(1+2\lambda^2)^2}\right] + \frac{\pi^2}{12}$$

$$\left. - \frac{2\lambda^2}{(1+2\lambda^2)} + \frac{\lambda^4}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6} + \frac{1}{4}\right)\right]\Bigg\}\Bigg]$$

$$\frac{E_1}{2Eabt} = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[H_1(\lambda)(1+\pi^2)^2 + H_2(\lambda)(1+\pi^2) + H_3(\lambda)\right] + \left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)\lambda^2\frac{\sigma}{E}\left\{\frac{1}{2\lambda^2} - \frac{3}{128}(1+\pi^2)\right\}$$

$$H_1(\lambda) = \frac{105}{32768} + \frac{9\lambda^4}{32768} + \frac{77\lambda^4}{65536} + \frac{(1-4\lambda^2)^2\lambda^4}{2048(1+8\lambda^2)^2} + \frac{(1-\lambda^2)^2\lambda^4}{32768(1+2\lambda^2)^2} + \frac{\lambda^4}{512(1+2\lambda^2)^2}$$

$$+ \frac{\lambda^4}{2048(2+\lambda^2)^2} + \frac{\lambda^6}{256(1+2\lambda^2)^2} + \frac{\lambda^6}{4096(2+\lambda^2)^2} + \frac{9\lambda^6}{1024(1+8\lambda^2)^2} + \text{———}$$

$$+ \frac{9\lambda^6}{65536(1+2\lambda^2)^2} + \frac{9\lambda^6}{1024(1+8\lambda^2)^2} + \frac{9\lambda^6}{65536(1+2\lambda^2)^2}$$

$$H_1(\lambda) = \frac{105}{32768} + \frac{95\lambda^4}{65536} + \frac{\lambda^4(1+2\lambda^2)}{2048(1+8\lambda^2)} + \frac{\lambda^4(130+\lambda^2)}{65536(1+2\lambda^2)} + \frac{3\lambda^6}{4096(2+\lambda^2)^2}$$

48б

$$H_2(\lambda) = -\frac{5}{48} - \frac{3}{256}\lambda^2 - \frac{\lambda^6}{32(1+2\lambda^2)^3} - \frac{\lambda^4}{64(1+2\lambda^2)^2} - \frac{\lambda^4}{16(1+2\lambda^2)^3} - \frac{\lambda^4}{96(2+\lambda^2)(1+2\lambda^2)} + \frac{\lambda^4}{64(1+2\lambda^2)(2+\lambda^2)^2}$$

$$+ \frac{\lambda^4(1+\lambda^2)}{16(1+2\lambda^2)^3} + \frac{\lambda^4}{16(1+2\lambda^2)^3} - \frac{\lambda^4}{16(1+2\lambda^2)} + \frac{\lambda^6}{32(1+2\lambda^2)^2} - \frac{\lambda^4}{64(1+2\lambda^2)(2+\lambda^2)^2} + \frac{\lambda^4}{128(2+\lambda^2)}$$

$$- \frac{\lambda^6}{48(1+2\lambda^2)(2+\lambda^2)}$$

$$\boxed{H_2(\lambda) = -\frac{5}{48} - \frac{3\lambda^2}{256} - \frac{3\lambda^4}{64(1+2\lambda^2)} - \frac{\lambda^6}{384(2+\lambda^2)}}$$

$$H_3(\lambda) = \frac{\pi^2}{8} - \frac{3}{16} + \frac{1}{8} + \frac{\lambda^4}{8(1+2\lambda^2)^2} + \frac{1}{16} + \frac{\lambda^4}{8(1+2\lambda^2)^3} + \frac{\lambda^4}{2(1+2\lambda^2)^4} - \frac{\lambda^2}{4(1+2\lambda^2)} - \frac{1}{8(1+2\lambda^2)}$$

$$+ \frac{1}{4}\frac{\lambda^4}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6} - \frac{1}{4}\right) + \frac{\lambda^2}{4(1+2\lambda^2)^3}(\sqrt{2}\lambda\pi)\coth\sqrt{2}\lambda\pi - \frac{\lambda^4}{(1+2\lambda^2)^4} + \frac{\pi^2\lambda^2\left(\frac{1}{2} + \frac{1}{4\sqrt{2}\pi\lambda}\sinh 2\sqrt{2}\lambda\pi\right)}{4(1+2\lambda^2)^2(\cosh 2\sqrt{2}\lambda\pi - 1)}$$

$$+ \frac{\lambda^2(1+\lambda^2)^2}{4(1+2\lambda^2)^4} + \frac{\lambda^2(1+\lambda^2)}{2(1+2\lambda^2)^4} - \frac{\lambda^2(1+\lambda^2)}{2(1+2\lambda^2)^2} + \frac{\lambda^4(1+\lambda^2)}{4(1+2\lambda^2)^3} + \frac{\pi^2\lambda^2\left(-\frac{1}{2} + \frac{1}{4\sqrt{2}\pi\lambda}\sinh 2\sqrt{2}\lambda\pi\right)}{4(1+2\lambda^2)^2(\cosh 2\sqrt{2}\lambda\pi - 1)}$$

$$- \frac{1}{8(1+2\lambda^2)}(\sqrt{2}\lambda\pi)\coth\sqrt{2}\lambda\pi + \frac{1}{8}\frac{1}{(1+2\lambda^2)} + \frac{\lambda^4}{2(1+2\lambda^2)^3}\sqrt{2}\pi\lambda\coth\sqrt{2}\lambda\pi + \frac{(1-2\lambda^2)\lambda^4}{2(1+2\lambda^2)^4} + \frac{\pi^2}{24}\lambda^2$$

$$- \frac{\lambda^4}{(1+2\lambda^2)} + \frac{\lambda^6}{2(1+2\lambda^2)^2}\left(\frac{\pi^2}{6} + \frac{1}{4}\right)$$

487

$$H_3(\lambda) = \frac{\pi^2}{8} + \frac{3\lambda^2(2-\lambda^2)}{8(1+2\lambda^2)^3} + \frac{\lambda^4(3+2\lambda^2)}{8(1+2\lambda^2)^3} - \frac{\lambda^2(1+4\lambda^2)}{4(1+2\lambda^2)} - \frac{\lambda^2(8+9\lambda^2-2\lambda^4)}{16(1+2\lambda^2)^2}$$

$$+ \frac{\pi^2}{24}\lambda^2\left[\frac{\lambda^2}{(1+2\lambda^2)}+1\right] + \left[\frac{\lambda^2}{4(1+2\lambda^2)^2} - \frac{1}{8(1+2\lambda^2)} + \frac{1}{16(1+2\lambda^2)^2}\right]\sqrt{2}\pi\lambda \coth\sqrt{2}\lambda\pi$$

$$H_3(\lambda) = \frac{\pi^2}{8} + \frac{\lambda^2(3+\lambda^4)}{4(1+2\lambda^2)^3} - \frac{3\lambda^2(4+11\lambda^2+10\lambda^4)}{16(1+2\lambda^2)^2} + \frac{\pi^2}{24}\lambda^2\left[\frac{\lambda^2}{(1+2\lambda^2)}+1\right]$$

$$- \frac{1}{16(1+2\lambda^2)^2}\sqrt{2}\pi\lambda \coth\sqrt{2}\lambda\pi$$

488

$$\frac{E_2}{Eabt} = \frac{1}{12}\left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{8}\right)^2\left[\frac{3}{8}+\frac{3}{8}\lambda^4+\frac{1}{4}\lambda^2\right] = \left(\frac{t}{R}\right)^2\left(\frac{f\pi^2}{8}\right)^2\left[\frac{1}{32}(1+\lambda^4)+\frac{1}{48}\lambda^2\right]$$

Thus total energy expression

$$= \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[H_1(\lambda)(f\pi^2)^2+H_2(\lambda)(f\pi^2)+H_3(\lambda)\right]+\left(\frac{t}{R}\right)^2\left(\frac{f}{8}\right)^2\left[\frac{1}{32}(1+\lambda^4)+\frac{1}{48}\lambda^2\right]\pi^4$$

$$-\left(\frac{a}{R}\right)^2\left(\frac{f}{8}\right)^2\pi^2\left(\frac{3\lambda^2}{8}\right)\frac{\sigma}{E} \quad -\frac{1}{3}\left(\frac{\sigma}{F}\right)^2$$

$$\therefore \quad \frac{\sigma}{E}\,\frac{3\pi^2\lambda^2}{4} = \left(\frac{a}{R}\right)^2\left[4H_1 f^2\pi^4+3H_2 f\pi^2+2H_3\right]+\frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\left[\frac{1}{16}(1+\lambda^4)+\frac{1}{24}\lambda^2\right]\pi^4$$

$$\lambda^2 K = \gamma^2\left[\frac{16}{3}H_1 f^2\pi^2+4H_2 f+\frac{8H_3}{3\pi^2}\right]+\frac{\pi^2}{\gamma^2}\left[\frac{1}{12}(1+\lambda^4)+\frac{1}{18}\lambda^2\right]$$

$$= \frac{\pi^2}{\gamma^2}\left[\frac{64}{3}H_1\left(\frac{f}{t}\right)^2+\left\{\frac{1}{12}(1+\lambda^4)+\frac{1}{18}\lambda^2\right\}\right]+\frac{8H_3}{3}\frac{\gamma^2}{\pi^2}+8H_2\left(\frac{f}{t}\right)$$

$$= 2\left[\frac{512}{9}H_1H_3\left(\frac{f}{t}\right)^2+H_3\left\{\frac{2}{9}(1+\lambda^4)+\frac{4}{27}\lambda^2\right\}\right]^{\frac{1}{2}}+8H_2\left(\frac{f}{t}\right)$$

489 ×

$$-a_2\frac{\sinh 2\pi}{2\pi}\left\{\left[\frac{1}{4(1+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2}+\frac{1}{20(1+4\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}\right](4\pi^2)+\left[1-\frac{1}{2(1+\lambda^2)}-\frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}\right]\right\}$$

$$+b_2\left[\frac{\sinh 2\pi}{2\pi}\left\{\left[\frac{1}{2}+\frac{\lambda^4}{2(1+\lambda^2)^2}+\frac{2}{5}\frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2}+\frac{(2\lambda^2+3)(6\lambda^2-1)}{100(1+4\lambda^2)^2}\right](4\pi^2)-\left[1+\frac{\lambda^4}{(1+\lambda^2)^2}\right]\right\}\right.$$

$$\left.-\cosh 2\pi\left\{\left[\frac{1}{4(1+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2}+\frac{1}{20(1+4\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}\right](4\pi^2)+\left[1-\frac{1}{2(1+\lambda^2)}-\frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}\right]\right\}\right]$$

$$-a_4\frac{\sinh 4\pi}{4\pi}(4\pi^2)\left\{\frac{1}{5(4+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}\right\}$$

$$+b_4(4\pi^2)\left[\frac{\sinh 4\pi}{4\pi}\left\{\frac{1}{8}+\frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2}-\frac{2\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}\right\}-\cosh 4\pi\left\{\frac{1}{5(4+\lambda^2)}+\frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}\right\}\right]$$

$$\cosh 2\pi\, a_2+2\pi\sinh 2\pi\, b_2=(4\pi^2)\left[\frac{1}{4}+\frac{1}{2(1+\lambda^2)^2}+\frac{1}{4(1+4\lambda^2)^2}\right]-\frac{1}{(1+\lambda^2)^2}$$

$$\sinh 2\pi\, a_2+(\sinh 2\pi+2\pi\cosh 2\pi)\,b_2=0$$

$$\frac{1}{2}(\sinh 4\pi+4\pi)b_2=-\sinh 2\pi\left[\qquad\right]$$

$$b_2=-\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi}+1}\left[(4\pi^2)\left\{\frac{1}{4}+\frac{1}{2(1+\lambda^2)^2}+\frac{1}{4(1+4\lambda^2)^2}\right\}-\frac{1}{(1+\lambda^2)^2}\right]$$

$$a_2=+\frac{\frac{\sinh 2\pi}{2\pi}+\cosh 2\pi}{\frac{\sinh 4\pi}{4\pi}+1}\left[\qquad\right]$$

489

$$a_4 = + \frac{\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} \quad f\, \pi^2 \left\{ \frac{1}{16} + \frac{1}{(4+\lambda^2)^2} \right\}$$

$$b_4 = - \frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} \quad f\, \pi^2 \left\{ \frac{1}{16} + \frac{1}{(4+\lambda^2)^2} \right\}$$

$$\frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} = \frac{42.613218}{11410.473} = 0.0037345707 \; ; \quad \frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} = 0.027199735$$

$$\frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} = 0.000006774685 5 \, , \quad \frac{\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} = 0.0000946 2 1164$$

Let us put

$$g_1(\lambda) = \frac{1}{4(1+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{4(1+\lambda^2)^2} + \frac{1}{20(1+4\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(1+4\lambda^2)^2}$$

$$g_2(\lambda) = 1 - \frac{1}{2(1+\lambda^2)} - \frac{\lambda^2(1-\lambda^2)}{2(1+\lambda^2)^2}$$

$$g_3(\lambda) = \frac{1}{2} + \frac{\lambda^4}{2(1+\lambda^2)^2} + \frac{2}{5}\,\frac{\lambda^2(\lambda^2-1)}{(1+4\lambda^2)^2} + \frac{(2\lambda^2+3)(6\lambda^2-1)}{100(1+4\lambda^2)^2}$$

$$g_4(\lambda) = 1 + \frac{\lambda^4}{(1+\lambda^2)^2}$$

$$g_5(\lambda) = \frac{1}{5(4+\lambda^2)} + \frac{\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}$$

$$g_6(\lambda) = \frac{1}{8} + \frac{(2+3\lambda^2)(6-\lambda^2)}{25(4+\lambda^2)^2} - \frac{2\lambda^2(1-\lambda^2)}{5(4+\lambda^2)^2}$$

$$H_1(\lambda) = \frac{1}{4} + \frac{1}{2(1+\lambda^2)^2} + \frac{1}{4(1+4\lambda^2)^2}$$

$$H_2(\lambda) = \frac{1}{(1+\lambda^2)^2}$$

$$H_3(\lambda) = \frac{1}{16} + \frac{1}{(4+\lambda^2)^2}$$

$$- \left[H_1 (f\pi^2) - H_2 \right] \left[\frac{\frac{\sinh 2\pi}{2\pi} + \cosh 2\pi}{\frac{\sinh 4\pi}{4\pi} + 1} \cdot \frac{\sinh 2\pi}{2\pi} \left(g_1 f\pi^2 + g_2 \right) + \frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} \frac{\sinh 2\pi}{2\pi} \left(g_3 f\pi^2 - g_4 \right) \right.$$

$$\left. - \frac{\frac{\sinh 2\pi}{2\pi}}{\frac{\sinh 4\pi}{4\pi} + 1} \cosh 2\pi \left(g_1 f\pi^2 + g_2 \right) \right]$$

$$- H_3 (f\pi^2)^2 \left[\frac{\frac{\sinh 4\pi}{4\pi} + \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} \frac{\sinh 4\pi}{4\pi} g_5 + \frac{\frac{\sinh 4\pi}{4\pi}}{\frac{\sinh 8\pi}{8\pi} + 1} \frac{\sinh 4\pi}{4\pi} g_6 - \frac{\frac{\sinh 4\pi}{4\pi} \cdot \cosh 4\pi}{\frac{\sinh 8\pi}{8\pi} + 1} g_5 \right]$$

$$= - \frac{\left(\frac{\sinh 2\pi}{2\pi} \right)^2}{\frac{\sinh 4\pi}{4\pi} + 1} \left[H_1 (f\pi^2) - H_2 \right] \left[\underbrace{(g_1 + g_3)}_{\frac{77}{100}} (f\pi^2) - \underbrace{(g_4 - g_2)}_{\frac{1}{2}} \right]$$

$\nearrow 0.15914286$

$0.12254000\ f\pi^2$

$- 0.07957143$

$$- \frac{\left(\frac{\sinh 4\pi}{4\pi} \right)^2}{\frac{\sinh 8\pi}{8\pi} + 1} H_3 (f\pi^2)^2 \left[\underbrace{g_5 + g_6}_{\frac{41}{200}} \right] = F_1$$

$\nearrow 0.07957 7486$

$0.01631338 4$

492

$$F_2 = \left[H_1(\theta\pi^2) - H_2\right]^2 \Big[\{(1+\lambda^2)^2\, 2\;11025500 + 0.00018495(1-\lambda^2)^2\} - \{0.28974167(3\lambda^4+2\lambda^2-1)$$

$$+ 3.64100103\,(1+\lambda^2)^2 - 0.00010158\lambda^{2}(1-\lambda^2)\} + \{0.03978203\big[4\pi^2(1+\lambda^2)^2 + \tfrac{1}{2}(5\lambda^4+2\lambda^2+1)\big] + 0.24995787(3\lambda^4+2\lambda^2-1)$$

$$+ 0.00003488\big[2(2\lambda^4-1) - \tfrac{4\pi^2}{3}(1-\lambda^2)^2\big]\}\Big]$$

$$+ \left[H_3(\theta\pi^2)\right]^2 \Big[3.66148735\,(1+\lambda^2)^2 - \{0.26989437(3\lambda^4+2\lambda^2-1) + 6.78318640(1+\lambda^2)^2\}$$

$$+ \{0.01989437\big[16\pi^2(1+\lambda^2)^2 + \tfrac{1}{9}(5\lambda^4+2\lambda^2+1)\big] + 0.2500000\,(3\lambda^4+2\lambda^2-1)\}\Big]$$

$$F_3 = \left\{\frac{17}{512}\left(1+\frac{1}{\lambda^4}\right) + \frac{1}{16(1+\lambda^2)^2} + \frac{1}{64(4+\lambda^2)^2} + \frac{1}{64(1+4\lambda^2)^2}\right\}(\theta\pi^2)^2$$

$$- \left\{\frac{1}{4\lambda^4} + \frac{1}{4(1+\lambda^2)^2}\right\} + \left\{\frac{1}{2\lambda^4} + \frac{1}{4(1+\lambda^2)^2}\right\}$$

413

$\lambda = 1.000$

$$F_3 = (0.0664 0625 + 0.015625 + 0.00125)(f\pi^2)^2 - (0.25 + 0.0625)(f\pi^2) + (0.5 + 0.0625)$$

$$= 0.08328125 (f\pi^2)^2 - 0.3125 (f\pi^2) + 0.5625$$

$H_1(\lambda) = 0.25 + 0.125 + 0.01 = 0.385$, $\quad H_2(\lambda) = 0.25$.

$H_3(\lambda) = 0.0625 + 0.04 = 0.1025$

$$F_1 = -[0.385(f\pi^2) - 0.25][0.12254000(f\pi^2) - 0.07957143] - 0.01631336 4 \times 0.1025 (f\pi^2)^2$$

$$= -0.04885002(f\pi^2)^2 + 0.06127000(f\pi^2) - 0.01989286$$

$$F_2 = [0.385(f\pi^2) - 0.25]^2 \, 0.15920493 + (0.1025 f\pi^2)^2 \, 0.07957575$$

$$= 0.02359815(f\pi^2)^2 - 0.03064695(f\pi^2) + 0.00995031 + 0.00083604(f\pi^2)^2$$

$$6\pi^2 \frac{\sigma}{E} = \left(\frac{a}{R}\right)^2 \left\{ 0.23546168 \pi^4 f^2 - 0.84563085 \pi^2 f + 0.55255745 \right\}_{\times 2} + \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2} \pi^2 \frac{16}{3}$$

$$K = \gamma^2 \left\{ 0.03924361 \pi^2 f^2 - 0.14093848 f + \frac{0.09209291}{\pi^2} \right\}^{\times 2} + \frac{1}{\gamma^2} \pi^2 \, 0.8888889$$

$$= \frac{\pi^2}{\gamma^2} \left\{ 0.15697444 \left(\frac{\delta}{t}\right)^2 + 0.88888889 \right\} + \frac{0.09209291 \gamma^2}{\pi^2}^{\times 2} - 0.28187696 \left(\frac{\delta}{t}\right)$$

495

$$K = 2\left\{ 0.02891247\left(\frac{\delta}{t}\right)^2 + 0.16372073 \right\}^{\frac{1}{2}} - 0.28187696\left(\frac{\delta}{t}\right)$$

$$K_0 = 0.8092$$

$$0.05782494\left(\frac{\delta}{t}\right) = 0.28187696\left\{ 0.02891247\left(\frac{\delta}{t}\right)^2 + 0.16372073 \right\}^{\frac{1}{2}}$$

$$0.00334372\left(\frac{\delta}{t}\right)^2 = 0.01300837, \qquad \left(\frac{\delta}{t}\right)^2 = \frac{0.01300837}{0.00104649} = 12.4305$$

$$-0.00229723$$

$$\left(\frac{\delta}{t}\right) = 3.5257$$

$$K_{min} = 0.4527$$

① (δ/t)	② $0.028912\left(\frac{\delta}{t}\right)^2$	③ ② $+0.16372$	④ 2 ③$^{\frac{1}{2}}$	⑤ K			
1	0.028912	0.19263	0.8772	0.5953			
2	0.115648	0.27937	1.0572	0.4934			
3	0.260208	0.42393	1.3020	0.4564			
4	0.462592	0.62631	1.5828	0.4553			
5	0.722800	0.88652	1.8832	0.4738			
6	1.040832	1.20455	2.1975	0.5062			
7	1.416688	1.58041	2.5142	0.5411			
8	1.850368	2.01409	2.8384	0.5834			
9	2.341872	2.50559	3.1658	0.6289			
10	2.89120	3.05492	3.4956	0.6768			

$\lambda = 1.200$

$$F_3 = \{0.05626085 + 0.01049785 + 0.000527986 + 0.00034192\}(f\pi^2)^2$$

$$- \{0.12056327 + 0.04199140\}(f\pi^2) + \{0.24112654 + 0.04199140\}$$

$$= 0.06762861\,(f\pi^2)^2 - 0.16255467\,(f\pi^2) + 0.28311794$$

$$H_1(\lambda) = 0.25 + 0.08398280 + 0.00547075 = 0.33945355$$

$$H_2(\lambda) = 0.16796560$$

$$H_3(\lambda) = 0.0625 + 0.03379109 = 0.09629109$$

$$F_1 = -[0.33945355\,(f\pi^2) - 0.16796560][0.12254000\,(f\pi^2) - 0.07957143] - 0.00157083\,(f\pi^2)^2$$

$$= -0.04316747\,(f\pi^2)^2 + 0.04759331\,(f\pi^2) - 0.01336526$$

$F_2 = [0.339[illegible]\,(f\pi^2) - 0.[illegible]]^2\,[0.19609603] + 0.0990[illegible]\,[0.0962[illegible]]^2 [illegible]$

$= 0.[illegible]\,[0.[illegible]\,(f\pi^2)^2 - 0.[illegible]\,(f\pi^2) + 0.[illegible]]$

$+ 0.00091800\,(f\pi^2)^2$

$$\pi^2 \frac{\sigma}{E} = \lambda^2 \left(\frac{a}{R}\right)^2 \{0.19282196\,\pi^4 f^2 - 0.41265255\,\pi^2 f + 0.550682900\}\frac{1}{6}$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\{0.93370370\}\pi^4$$

$$K = \beta^2\left\{0.04627727\,\pi^2 f^2 - 0.09903661 f + \frac{0.13216390}{\pi^2}\right\} + \frac{0.93370370\,\pi^2}{\beta^2}$$

497

$$K = \frac{\pi^2}{g^2}\left\{0.18510908\left(\frac{\delta}{t}\right)^2 + 0.93370370\right\} + \frac{0.13216390}{\pi^2}g^2 - 0.19807322\left(\frac{\delta}{t}\right)$$

$$= 2\left\{0.02446474\left(\frac{\delta}{t}\right)^2 + 0.12340192\right\}^{\frac{1}{2}} - 0.19807322\left(\frac{\delta}{t}\right)$$

$K_0 = 0.7026$

$$0.04892948\left(\frac{\delta}{t}\right) = 0.19807322\left\{0.02446474\left(\frac{\delta}{t}\right)^2 + 0.12340192\right\}$$

$$0.002394094\left(\frac{\delta}{t}\right)^2 = 0.00484143$$

0.000959825

$$\left(\frac{\delta}{t}\right)^2 = 3.325538$$

$$\left(\frac{\delta}{t}\right) = 1.837264$$

$K_{min.} = 0.5438$

①	②	③	④	⑤	⑥	⑦
(δ/t)	0.024464 ①^2	② $+0.12340$	$2\times$ ③$^{\frac{1}{2}}$	K		
1	0.024464	0.14786	0.7690	0.5709		
2	0.097856	0.22126	0.9408	0.5447		
3	0.220176	0.34358	1.1724	0.5782		
4	0.391424	0.51482	1.4350	0.6422		
5	0.611600	0.73500	1.7146	0.7243		

$\lambda = 0.8000$

$$F_3 = \left\{ 0.11426544 + 0.02323766 + 0.000725745 + 0.00123288 \right\} (f\pi^2)^2$$

$$- \left\{ 0.61035156 + 0.09295062 \right\} (f\pi^2) + \left\{ 1.22070313 + 0.09295062 \right\}$$

$$= 0.13946173 (f\pi^2)^2 - 0.70330218 (f\pi^2) + 1.31365375$$

$$H_1(\lambda) = 0.25 + 0.18590125 + 0.01972604 = 0.45562729$$

$$H_2(\lambda) = 0.37180250 , \quad H_3(\lambda) = 0.0625 + 0.04644768 = 0.10894768$$

$$-\left[0.45562729 (f\pi^2) - 0.37180250\right]\left[0.1225400 (f\pi^2) - 0.07957143\right]$$

$$- 0.016313384 \times 0.10894768 (f\pi^2)^2$$

$$= - 0.05760987 (f\pi^2)^2 + 0.08181559 (f\pi^2) - 0.02961491 = F_4$$

$$F_2 = 0.13302164 \left[0.45562729 (f\pi^2) - 0.37180250 \right]^2$$

$$+ 0.0654153 \times (0.10894768)^2 (f\pi^2)^2$$

$$= 0.13302164 \left\{ 0.20759623 (f\pi^2)^2 - 0.33880673 (f\pi^2) + 0.13823710 \right\}$$

$$+ 0.00157891 (f\pi^2)^2$$

$$\pi^2 \frac{\sigma}{E} = \left(\frac{a}{R}\right)^2 \left\{ 0.28427664 \pi^4 f^2 - 1.27978602 \pi^2 f + 1.66710703 \right\} \frac{1}{6}$$

$$+ \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2} \left\{ 0.95638889 \right\} \pi^4$$

$$K = \rho^2 \left\{ 0.04737944\, \pi^2 f^2 - 0.21329767 f + \frac{0.277851172}{\pi^2} \right\} + \frac{\pi^2}{\rho^2} 0.95638889 \qquad \underline{499}$$

$$= \frac{\pi^2}{\rho^2} \left\{ 0.18951776 \left(\frac{\delta}{t}\right)^2 + 0.95638889 \right\} + \frac{0.277851172\, \rho^2}{\pi^2} - 0.42659534 \left(\frac{\delta}{t}\right)$$

$$= 2 \left\{ 0.05265773 \left(\frac{\delta}{t}\right)^2 + 0.26573377 \right\}^{\frac{1}{2}} - 0.42659534 \left(\frac{\delta}{t}\right)$$

$$K_0 = 1.0310$$

$$0.10531546 \left(\frac{\delta}{t}\right) = 0.42659534 \left(0.05265773 \left(\frac{\delta}{t}\right)^2 + 0.26573377\right)^{\frac{1}{2}}$$

$$\begin{array}{r} 0.011091346 \\ 0.009582842 \\ \hline 0.001508504 \end{array} \left(\frac{\delta}{t}\right)^2 = 32.05771 \qquad \left(\frac{\delta}{t}\right) = 5.66195$$

$$K_{min} = 0.3802$$

①	②	③	④	⑤	⑥		
(δ/t)							
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							

$\lambda = 0.600$

$$F_3 = \{0.28940008 + 0.03379109 + 0.00082195 + 0.00262446\}(f\pi^2)^2$$

$$- \{1.92901235 + 0.13516436\}(f\pi^2) + \{3.85802469 + 0.13516436\}$$

$$= 0.32663758\,(f\pi^2)^2 - 2.06417671\,(f\pi^2) + 3.99318905$$

$$H_1(\lambda) \doteq 0.25 + 0.27032872 + 0.04199140 = 0.56232012, \quad H_2(\lambda) = 0.54065744$$

$$H_3(\lambda) = 0.0625 + 0.05260500 = 0.11510500$$

$$F_1 = -[0.56232012\,(f\pi^2) - 0.54065744][0.12254000\,(f\pi^2) - 0.07957143]$$

$$- 0.11510500 \times 0.01631338\,4\,(f\pi^2)^2$$

$$= -0.07078446\,(f\pi^2)^2 + 0.11099678\,(f\pi^2) - 0.04302089$$

$$F_2 = 0.11619690\,[0.56232012\,(f\pi^2) - 0.54065744]^2$$

$$+ (0.11510500)^2 (f\pi^2)^2 \times 0.05818125$$

$$= 0.11619690\,[0.31620392\,(f\pi^2)^2 - 0.60804511\,(f\pi^2) + 0.2923\text{[illegible]}] + 0.00077092\,(f\pi^2)^2$$

$$K = \delta^2\left\{0.07040783\,\pi^2 f^2 - 0.36428992\,f + \frac{0.47809605}{\pi^2}\right\} + \frac{\pi^2}{\delta^2}\,1.26814815$$

$$= \frac{\pi^2}{\delta^2}\left\{0.28163132\left(\frac{\delta}{t}\right)^2 + 1.26814815\right\} + \frac{0.47809605\,\delta^2}{\pi^2} - 0.72857984\left(\frac{\delta}{t}\right)$$

$$= 2\left\{0.13464682\left(\frac{\delta}{t}\right)^2 + 0.60629662\right\}^{\frac{1}{2}} - 0.72857984\left(\frac{\delta}{t}\right)$$

$$K_0 = 1.55730$$

$$0.26929364\left(\frac{\delta}{t}\right) = 0.72857984\left\{0.13464682\left(\frac{\delta}{t}\right)^2 + 0.60629662\right\}^{\frac{1}{2}} \qquad \underline{\underline{501}}$$

$$\begin{array}{r} 0.072519065 \\ \underline{0.071474380} \\ 0.001044685 \end{array} \bigg) \left(\frac{\delta}{t}\right)^2 = 308.0733 \qquad \left(\frac{\delta}{t}\right) = 17.55202$$

$$\underline{\underline{K_{min} = 0.1869}}$$

①	②	③	④	⑤	⑥		
(δ/t)							
2							
4							
6							
8							
10							
12							
14							
16							
18							
20							
22							

502

Consider the limiting case, when $\lambda << 1$.

then $$6\pi^2 \frac{\sigma}{E} = \frac{1}{\lambda^2}\left[\frac{17}{128} f^2 \pi^4 - \frac{3}{4} f \pi^2 + 1\right]\left(\frac{a}{R}\right)^2 \frac{\left(\frac{t}{R}\right)^2}{\left(\frac{a}{R}\right)^2}\, 2\frac{1}{\lambda^2}\,\pi^4$$

$$\text{or} \quad \lambda^2 K = \gamma^2\left[\frac{17}{768}\pi^2 f^2 - \frac{1}{8} f + \frac{1}{6\pi^2}\right] + \frac{\pi^2}{\gamma^2}\frac{1}{3}$$

$$= \frac{\pi^2}{\gamma^2}\left[\frac{17}{192}\left(\frac{\delta}{t}\right)^2 + \frac{1}{3}\right] + \frac{1}{6}\frac{\gamma^2}{\pi^2} - \frac{1}{4}\left(\frac{\delta}{t}\right)$$

$$= 2\left[\frac{17}{1152}\left(\frac{\delta}{t}\right)^2 + \frac{1}{18}\right]^{\frac{1}{2}} - \frac{1}{4}\left(\frac{\delta}{t}\right) \qquad \underline{\text{O.K.}}$$

$$\frac{17}{1152}\left(\frac{\delta}{t}\right)^2 + \frac{1}{18} = \frac{1}{64}\left(\frac{\delta}{t}\right)^2$$

$$\frac{1}{1152}\left(\frac{\delta}{t}\right)^2 = \frac{1}{18} \qquad \left(\frac{\delta}{t}\right) = \sqrt{\frac{1152}{18}} = 8$$

$$\gamma^2_{min} = \pi^2\left[34 + 2\right]^{\frac{1}{2}} = \pi^2 36 \qquad \gamma = 6^{\frac{1}{2}}\pi$$

$\lambda = 0.4$

$$F_3 = \left\{ 1.33020020 + 0.04644768 + 0.00090289 + 0.00580941 \right\} (f\pi^2)^2$$
$$- \left\{ 9.76562500 + 0.18579073 \right\} (f\pi^2) + \left\{ 19.53125000 + 0.18579073 \right\}$$
$$= 1.38336018\, (f\pi^2)^2 - 9.95141573\, (f\pi^2) + 19.71704073$$

$H_1(\lambda) = 0.25 + 0.37158145 + 0.09295062 = 0.71453207$, $\quad H_2(\lambda) = 0.74316290$

$H_3(\lambda) = 0.0625 + 0.05778476 = 0.12028476$

$$F_1 = -\left[0.71453207 (f\pi^2) - 0.74316290 \right] \left[0.1225400 (f\pi^2) - 0.07957143 \right]$$
$$- 0.12028426 \times 0.016313384\, (f\pi^2)^2$$
$$= -0.08952101\, (f\pi^2)^2 + 0.14792352\, (f\pi^2) - 0.05913453$$

$$F_2 = 0.09004850 \left[0.71453207 (f\pi^2) - 0.74316290 \right]^2 + 0.04515168 \times (0.12028476)^2 (f\pi^2)^2$$
$$= 0.09004850 \left[0.51055608\, (f\pi^2)^2 - 1.0620272\, (f\pi^2) + 0.5522910 \right]$$
$$+ 0.00065327\, (f\pi^2)^2$$

$$K = \gamma^2 \left\{ 0.142983\, \pi^2 f^2 - 0.79193010 f + \frac{1.05107409}{\pi^2} \right\} + \frac{\pi^2}{\gamma^2}\, 2.35888889$$
$$= \frac{\pi^2}{\gamma^2} \left\{ 0.57193268 \left(\frac{\delta}{t}\right)^2 + 2.35888889 \right\} + \frac{1.05107409\, \gamma^2}{\pi^2} - 1.58386020 \left(\frac{\delta}{t}\right)$$
$$= 2 \left\{ 0.60114362 \left(\frac{\delta}{t}\right)^2 + 2.47936199 \right\}^{\frac{1}{2}} - 1.58386020 \left(\frac{\delta}{t}\right)$$

$K_0 = 3.14920,$

$$\gamma^2 = \pi^2 \{$$

~~1.20228724 $\left(\frac{\delta}{t}\right)$ = 1.58[illegible] $\{0.60114362 \left(\frac{\delta}{t}\right)^2 + 2.$[illegible]$\}^{\frac{1}{2}}$~~

1.~~44549461~~

$$0.60114362 \left(\frac{\delta}{t}\right)^2 + 2.47936699 = 0.62715328 \left(\frac{\delta}{t}\right)^2$$

$$\begin{array}{r} 0.60114362 \\ \hline 0.02600966 \end{array}$$

$$\left(\frac{\delta}{t}\right)_0 = 9.7588$$

JOURNAL OF THE AERONAUTICAL SCIENCES

20th and Northampton Streets, Easton, Pa.

Postmaster: Please send undeliverable copies and notices to

5111 RCA Building
Rockefeller Center
New York, N. Y.

RETURN POSTAGE GUARANTEED

Printed in United States of America

Entered as second class mail matter at the Easton, Pa., Post Office

Shell II

Mr. H. T. Fan,
Aeronautics Dept.,
California Institute of Tech.,
Pasadena, California CT

$$\frac{w}{R}=\frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2-\frac{f}{4}\left(1+\cos\frac{\pi x}{a}\right)\left(1+\cos\frac{\pi y}{b}\right)\right]$$

$$\frac{w_0}{R}=\frac{1}{2}\left(\frac{a}{R}\right)^2\left[1-\left(\frac{x}{a}\right)^2\right]$$

$$\sigma_x=E\left\{\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2-\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right\}$$

$$\sigma_y=E\left\{\frac{\partial v}{\partial y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\right\}$$

$$\tau_{xy}=E\left\{\frac{1}{2}\frac{\partial v}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right\}$$

By using the equilibrium equation: $\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0$, we have

$$\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=-\frac{1}{2}\left\{\frac{\partial w}{\partial y}\left(\frac{\partial^2 w}{\partial x^2}+2\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x\partial y}\right\}$$

We have

$$\frac{\partial w}{\partial x}=\left(\frac{a}{R}\right)\left\{-\left(\frac{x}{a}\right)+\frac{f}{8}\pi\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right\}$$

$$\frac{\partial w}{\partial y}=\left(\frac{a}{R}\right)\frac{f}{8}\pi\left(\frac{a}{b}\right)\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}$$

$$\frac{\partial^2 w}{\partial x^2}=\frac{1}{R}\left[-1+\frac{f}{8}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)\right\}$$

$$\frac{\partial^2 w}{\partial y^2}=\frac{1}{R}\left[\frac{f}{8}\pi^2\left(\frac{a}{b}\right)^2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}\right]$$

$$\frac{\partial^2 w}{\partial x\partial y}=\frac{1}{R}\left[-\frac{f}{8}\pi^2\left(\frac{a}{b}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$R\left[\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right]=-\left(\frac{a}{R}\right)\frac{f}{g}\pi\lambda\left[\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}\left\{\frac{f}{g}\pi^2\cos\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)+\frac{f}{g}\pi^2\left(\frac{a}{b}\right)^2 2\left(1+\cos\frac{\pi x}{a}\right)\cos\frac{\pi y}{b}-1\right\}\right.$$

$$\left.-\pi\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\left\{\frac{f}{g}\pi\sin\frac{\pi x}{a}\left(1+\cos\frac{\pi y}{b}\right)-\frac{x}{a}\right\}\right]$$

$$=-\left(\frac{a}{R}\right)\frac{f}{g}\pi\lambda\left[\frac{f\pi^2}{\cancel{32}\,16}\left(\cancel{1}+\cancel{2}\cos\frac{\pi x}{a}+\cancel{1}\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)+\frac{f\pi^2}{16}\left(\frac{a}{b}\right)^2\left(3+4\cos\frac{\pi x}{a}+\cos\frac{2\pi x}{a}\right)\sin\frac{2\pi y}{b}\right.$$

$$-\left(1+\cos\frac{\pi x}{a}\right)\sin\frac{\pi y}{b}$$

$$\left.-\cancel{\frac{f\pi^2}{32}\left(1-\cos\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b}+\sin\frac{2\pi y}{b}\right)}+\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$=-\left(\frac{a}{R}\right)\frac{f}{g}\pi\lambda\left[\frac{f\pi^2}{8}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{8}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{16}\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}\right.$$

$$+\frac{f\pi^2}{16}\left(\frac{a}{b}\right)^2 3\sin\frac{2\pi y}{b}+\frac{f\pi^2}{4}\left(\frac{a}{b}\right)^2\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}\left(\frac{a}{b}\right)^2\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$\left.-\sin\frac{\pi y}{b}-\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

470

$$R\left[\frac{\partial^2 v}{\partial x^2}+2\frac{\partial^2 v}{\partial y^2}\right] = -\left(\frac{q}{R}\right)\frac{f}{8}\pi\lambda\left[-\sin\frac{\pi y}{b}+\frac{3}{16}(f\pi^2)\lambda^2\sin\frac{2\pi y}{b}+\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}\right.$$

$$+\frac{f\pi^2}{8}\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}+\frac{f\pi^2}{16}(1+4\lambda^2)\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}+\frac{f\pi^2}{16}(1+\lambda^2)\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$\left.+\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

The particular integral is

$$R\,v = +\left(\frac{q}{R}\right)\frac{f}{8}\pi\lambda\left[-\frac{\sin\frac{\pi y}{b}}{2\left(\frac{\pi}{b}\right)^2}+\frac{3}{16}(f\pi^2)\lambda^2\frac{\sin\frac{2\pi y}{b}}{8\left(\frac{\pi}{b}\right)^2}+\left(\frac{f\pi^2}{8}-1\right)\frac{\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\right.$$

$$+\frac{f\pi^2}{8}\frac{\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}}{4\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}+\frac{f\pi^2}{16}(1+4\lambda^2)\frac{\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b}}{\left(\frac{\pi}{a}\right)^2+8\left(\frac{\pi}{b}\right)^2}+\frac{f\pi^2}{16}(1+\lambda^2)\frac{\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}}{4\left(\frac{\pi}{a}\right)^2+8\left(\frac{\pi}{b}\right)^2}$$

$$\left.+\frac{2\left(\frac{\pi}{a}\right)^2}{\left[\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2\right]^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}+\frac{1}{\left(\frac{\pi}{a}\right)^2+2\left(\frac{\pi}{b}\right)^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\right]$$

$$\frac{v}{R} = \left(\frac{a}{R}\right)^3 \frac{f}{g}\frac{1}{\pi}\lambda\Big[-\frac{1}{2\lambda^2}\sin\frac{\pi y}{b} + \frac{3}{128}(f\pi^2)\sin\frac{2\pi y}{b} + \left(\frac{f\pi^2}{g}-1\right)\frac{\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}}{1+2\lambda^2}$$

$$+ \frac{f\pi^2}{16}\frac{\cos\frac{2\pi x}{a}\sin\frac{\pi y}{b}}{2+\lambda^2} + \frac{f\pi^2}{16}\left(\frac{1+4\lambda^2}{1+8\lambda^2}\right)\cos\frac{\pi x}{a}\sin\frac{2\pi y}{b} + \frac{f\pi^2}{64}\left(\frac{1+\lambda^2}{1+2\lambda^2}\right)\cos\frac{2\pi x}{a}\sin\frac{2\pi y}{b}$$

$$+ \frac{2\lambda^2}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + a_0\left(\frac{\pi y}{b}\right) + a_1\cosh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}\Big]$$

$$\frac{\partial v}{\partial x} = \left(\frac{a}{R}\right)^2 \frac{f}{g}\lambda\Big[-\left\{\frac{f\pi^2}{g}(1+2\lambda^2)-1\right\}\frac{\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}}{(1+2\lambda^2)^2} - \frac{f\pi^2}{g}\frac{\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b}}{(2+\lambda^2)} - \frac{f\pi^2}{16}\left(\frac{1+4\lambda^2}{1+8\lambda^2}\right)\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}$$

$$- \frac{f\pi^2}{32}\left(\frac{1+\lambda^2}{1+2\lambda^2}\right)\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{b} + \frac{1}{1+2\lambda^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{1}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$+ a_1\sqrt{2}\lambda\sinh\frac{\sqrt{2}\lambda\pi x}{a}\sin\frac{\pi y}{b}\Big]$$

$$\boxed{a_1 = \frac{\pi}{(1+2\lambda^2)\sqrt{2}\lambda\sinh\sqrt{2}\lambda\pi} = \frac{1}{2\lambda^2(1+2\lambda^2)}\frac{1}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2 \frac{f}{8}\Big[-\frac{1}{2}\cos\frac{\pi y}{b} + \frac{3\lambda^2}{64}(f\pi^2)\cos\frac{2\pi y}{b} + \frac{\lambda^2}{1+2\lambda^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

$$+ \frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}\cos\frac{\pi y}{b} + \frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2}\frac{f\pi^2}{8}\cos\frac{\pi x}{a}\cos\frac{2\pi y}{b} + \frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2}\frac{f\pi^2}{32}\cos\frac{2\pi x}{a}\cos\frac{2\pi y}{b}$$

$$+ \frac{2\lambda^4}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a}\cos\frac{\pi y}{b} + \frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}\cos\frac{\pi y}{b} + a_0\lambda^2 + a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\cos\frac{\pi y}{b}\Big]$$

$$\left(\frac{\partial v}{\partial y}\right)_{y=b} = \left(\frac{a}{R}\right)^2\frac{f}{8}\Big[\frac{1}{2} + \frac{3\lambda^2}{64}(f\pi^2) - \frac{\lambda^2}{1+2\lambda^2}\left(\frac{f\pi^2}{8}-1\right)\cos\frac{\pi x}{a} - \frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\cos\frac{2\pi x}{a}$$

$$+ \frac{\lambda^2(1+4\lambda^2)}{1+8\lambda^2}\frac{f\pi^2}{8}\cos\frac{\pi x}{a} + \frac{\lambda^2(1+\lambda^2)}{1+2\lambda^2}\frac{f\pi^2}{32}\cos\frac{2\pi x}{a} - \frac{2\lambda^4}{(1+2\lambda^2)^2}\cos\frac{\pi x}{a} - \frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{a}\right)$$

$$+ a_0\lambda^2 - a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\Big]$$

$$-\frac{\sigma}{E} = \left(\frac{a}{R}\right)^2\frac{f}{8}\Big[\frac{1}{2} + \frac{3\lambda^2}{64}(f\pi^2) - \frac{\lambda^2}{1+2\lambda^2} + a_0\lambda^2 - \frac{a_1\lambda^2}{\sqrt{2}\lambda\pi}\sinh\sqrt{2}\lambda\pi\Big]$$

$$= \left(\frac{a}{R}\right)^2\frac{f}{8}\Big[\frac{1}{2} + \frac{3}{64}\lambda^2(f\pi^2) - \frac{\lambda^2}{1+2\lambda^2} + a_0\lambda^2 - \frac{1}{2(1+2\lambda^2)}\Big]$$

$$= \left(\frac{a}{R}\right)^2\frac{f}{8}\Big[\frac{3}{64}\lambda^2(f\pi^2) + a_0\lambda^2\Big]$$

$$\left(\frac{a}{R}\right)^2 \frac{f}{8} a_0 \lambda^2 = -\frac{\sigma}{E} - \left(\frac{a}{R}\right)^2 \frac{f}{8}\ \frac{3}{64} \lambda^2 (f\pi^2)$$

$$\boxed{\frac{\Delta p}{E a b t} = -\left(\frac{\sigma}{E}\right)^2 - \left(\frac{a}{R}\right)^2 \left(\frac{f}{8}\right)^2 \left(\frac{3}{8} \lambda^2 \pi^2\right) \frac{\sigma}{E}}$$

$$\frac{\partial v}{\partial y} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \Big[-\frac{1}{2} \cos\frac{\pi y}{b} + \frac{3}{64} \lambda^2 (f\pi^2) \cos\frac{2\pi y}{b} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{f\pi^2}{8} - 1\right) \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}$$

$$+ \frac{\lambda^2}{9+\lambda^2} \frac{f\pi^2}{16} \cos\frac{2\pi x}{a} \cos\frac{\pi y}{b} + \frac{\lambda^2 (1+4\lambda^2)}{1+8\lambda^2} \frac{f\pi^2}{8} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} + \frac{\lambda^2 (1+\lambda^2)}{1+2\lambda^2} \frac{f\pi^2}{32} \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b}$$

$$+ \frac{2\lambda^4}{(1+2\lambda^2)^2} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} \cos\frac{\pi y}{b} - \frac{3}{64} \lambda^2 (f\pi^2) + a_1 \lambda^2 \cosh\frac{\sqrt{2}\lambda \pi x}{a} \cos\frac{\pi y}{b} \Big] - \frac{\sigma}{E}$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = \left(\frac{a}{R}\right)^2 \frac{f}{8} \Big[\frac{f\pi^2}{64} \lambda^2 \Big\{ 3 + 4\cos\frac{\pi x}{a} + \cos\frac{2\pi x}{a} - 3\cos\frac{2\pi y}{b} - 4\cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} - \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b} \Big\} \Big]$$

$$\frac{\sigma_y}{E} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \Big[\Big\{ \frac{f\pi^2}{16} \lambda^2 \cos\frac{\pi x}{a} + \frac{f\pi^2}{64} \lambda^2 \cos\frac{2\pi x}{a} \Big\}$$

$$+ \Big\{ -\frac{1}{2} + \left(\frac{f\pi^2}{8} \frac{\lambda^2}{1+2\lambda^2} - \frac{\lambda^2}{(1+2\lambda^2)^2} \right) \cos\frac{\pi x}{a} + \frac{\lambda^2}{9+\lambda^2} \frac{f\pi^2}{16} \cos\frac{2\pi x}{a} + \frac{\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \sin\frac{\pi x}{a} + a_1 \lambda^2 \cosh\frac{\sqrt{2}\lambda \pi x}{a} \Big\} \cos\frac{\pi y}{b}$$

$$+ \Big\{ \frac{\lambda^2}{1+8\lambda^2} \frac{f\pi^2}{16} \cos\frac{\pi x}{a} + \frac{\lambda^2}{1+2\lambda^2} \frac{f\pi^2}{64} \cos\frac{2\pi x}{a} \Big\} \cos\frac{2\pi y}{b} \Big] - \frac{\sigma}{E}$$

(4.24)

$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{h}\right)^4\left(\frac{f}{8}\right)^2\Bigg[\frac{1}{4}\left(\frac{f\pi^2\lambda^2}{16}\right)^2+\left(\frac{f\pi^2}{64}\lambda^2\right)^2\frac{1}{4}+\frac{1}{8}\left(\frac{\lambda^2}{1+8\lambda^2}\frac{f\pi^2}{16}\right)^2+\frac{1}{8}\left(\frac{\lambda^2}{1+2\lambda^2}\frac{f\pi^2}{64}\right)^2$$

$$+\frac{1}{16}+\frac{1}{8}\left(\frac{f\pi^2}{8}\frac{\lambda^2}{1+2\lambda^2}-\frac{\lambda^2}{(1+2\lambda^2)^2}\right)^2+\frac{1}{8}\left(\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\right)^2$$

$$-\frac{1}{4}\int_0^1\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{2}\left\{\frac{f\pi^2}{8}\frac{\lambda^2}{1+2\lambda^2}-\frac{\lambda^2}{(1+2\lambda^2)^2}\right\}\int_0^1\cos\frac{\pi x}{a}\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{2}\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{16}\int_0^1\cos\frac{2\pi x}{a}\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{1}{4}\int_0^1\left\{\frac{\lambda^2}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\sin\frac{\pi x}{a}+a_1\lambda^2\cosh\frac{\sqrt{2}\lambda\pi x}{a}\right\}^2 d\left(\frac{x}{a}\right)\Bigg]$$

445

$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_y^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\Bigg[\frac{17}{16384}(f\pi^2)^2\lambda^4 + \frac{1}{2048}\left(\frac{\lambda^2}{1+8\lambda^2}\right)^2(f\pi^2)^2 + \frac{1}{32768}\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2(f\pi^2)^2 + \frac{1}{16}$$

$$+\frac{1}{512}\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2(f\pi^2)^2 + \frac{1}{2048}\left(\frac{\lambda^2}{2+\lambda^2}\right)^2(f\pi^2)^2 - \frac{1}{32}\frac{\lambda^4}{(1+2\lambda^2)^3}(f\pi^2) + \frac{1}{8}\frac{\lambda^4}{(1+2\lambda^2)^4}$$

$$-\frac{1}{4}\frac{\lambda^2}{1+2\lambda^2} - \frac{1}{4}\frac{1}{2(1+2\lambda^2)} + \left\{\frac{f\pi^2}{8}\frac{\lambda^2}{1+2\lambda^2} - \frac{\lambda^2}{(1+2\lambda^2)^2}\right\}\left\{-\frac{1}{8}\frac{\lambda^2}{1+2\lambda^2} - \frac{1}{2}\frac{\lambda^2}{(1+2\lambda^2)^2}\right\}$$

$$+\frac{\lambda^2}{2+\lambda^2}\frac{f\pi^2}{32}\left\{-\frac{1}{3}\frac{\lambda^2}{1+2\lambda^2} + \frac{\lambda^2}{(1+2\lambda^2)(2+\lambda^2)}\right\} + \frac{1}{4}\left(\frac{\pi^2}{6}-\frac{1}{4}\right)\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2$$

$$+\frac{\lambda^2}{8(1+2\lambda^2)^2}\frac{1}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)}\left\{\frac{\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2} - \frac{4\lambda^2}{(1+2\lambda^2)^2}\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)\right\}$$

$$+\frac{1}{4}\frac{1}{4(1+2\lambda^2)^2}\frac{1}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)^2}\left\{\frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi\right\}\Bigg] - \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$\frac{1}{2E^2ab}\int_0^a\int_0^b \sigma_x^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{105}{32768}(f\pi^2)^2 - \frac{5}{48}(f\pi^2) + \left(\frac{\pi^2}{8}-\frac{3}{16}\right)\right]$$

476

$$\frac{\partial v}{\partial x} = \left(\frac{a}{R}\right)^2 \frac{f}{8} \lambda \Bigg[\left\{ 2 \frac{(1+\lambda^2)}{(1+2\lambda^2)^2} - \frac{1}{1+2\lambda^2} \frac{f\pi^2}{8} \right\} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} - \frac{1}{8(2+\lambda^2)} (f\pi^2) \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b}$$

$$- \left(\frac{1+4\lambda^2}{1+8\lambda^2}\right) \frac{f\pi^2}{16} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} - \left(\frac{1+\lambda^2}{1+2\lambda^2}\right) \frac{f\pi^2}{32} \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{b} + \frac{1}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \cos\frac{\pi x}{a} \sin\frac{\pi y}{b}$$

$$+ a_1 \sqrt{2} \lambda \sinh\frac{\sqrt{2}\lambda \pi x}{a} \sin\frac{\pi y}{b} \Bigg]$$

$$\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} = \left(\frac{a}{R}\right)^2 \frac{f\lambda}{8} \left\{ - \left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b} - \left(\frac{\pi x}{a}\right) \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{f\pi^2}{32} \left(2\sin\frac{\pi x}{a} + \sin\frac{2\pi x}{a}\right)\left(2\sin\frac{\pi y}{b} + \sin\frac{2\pi y}{b}\right) \right\}$$

$$= \left(\frac{a}{R}\right)^2 \frac{f\lambda}{8} \left\{ - \left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b} - \left(\frac{\pi x}{a}\right) \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{f\pi^2}{8} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \frac{f\pi^2}{16} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} \right.$$

$$\left. + \frac{f\pi^2}{16} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} + \frac{f\pi^2}{32} \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{b} \right\}$$

$$\frac{2\tau_{xy}}{E} = \left(\frac{a}{R}\right)^2 \frac{f\lambda}{8} \left\{ - \left(\frac{\pi x}{a}\right) \sin\frac{\pi y}{b} - \frac{2\lambda^2}{1+2\lambda^2} \left(\frac{\pi x}{a}\right) \cos\frac{\pi x}{a} \sin\frac{\pi y}{b} + 2 \left\{ \frac{1+\lambda^2}{(1+2\lambda^2)^2} + \frac{\lambda^2}{1+2\lambda^2} \frac{f\pi^2}{8} \right\} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \right.$$

$$+ \frac{\lambda^2}{16(2+\lambda^2)} (f\pi^2) \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b} + \frac{\lambda^2}{4(1+8\lambda^2)} (f\pi^2) \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} + \frac{\lambda^2}{(1+2\lambda^2)} \frac{f\pi^2}{32} \sin\frac{2\pi x}{a} \sin\frac{2\pi y}{b}$$

$$\left. + \frac{a_1}{2} \sqrt{2} \lambda \sinh\frac{\sqrt{2}\lambda \pi x}{a} \sin\frac{\pi y}{b} \right\}$$

477

$$\frac{\tau_{xy}}{E}=\left(\frac{a}{R}\right)^2\frac{f}{8}\left[\left[\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right)-\frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}+\frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}+\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2}+\frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\sin\frac{\pi x}{a}+\frac{\lambda^3}{32(2+\lambda^2)}(f\pi^2)\sin\frac{2\pi x}{a}\right]\sin\frac{\pi y}{b}\right.$$

$$\left.+\left\{\frac{\lambda^3}{8(1+8\lambda^2)}(f\pi^2)\sin\frac{\pi x}{a}+\frac{\lambda^3}{(1+2\lambda^2)}\frac{f\pi^2}{64}\sin\frac{2\pi x}{a}\right\}\sin\frac{2\pi y}{b}\right]$$

$$\frac{1}{E^2ab}\int_0^a\int_0^b\tau_{xy}^2\,dx\,dy=\left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\left[\frac{1}{4}\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2}+\frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}^2+\frac{1}{4}\left\{\frac{\lambda^3}{32(2+\lambda^2)}f\pi^2\right\}^2\right.$$

$$+\frac{1}{4}\left\{\frac{\lambda^3}{8(1+8\lambda^2)}f\pi^2\right\}^2+\frac{1}{4}\left\{\frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{64}\right\}^2$$

$$+\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2}+\frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\int_0^1\sin\frac{\pi x}{a}\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right)-\frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}+\frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$+\frac{\lambda^3}{32(2+\lambda^2)}(f\pi^2)\int_0^1\sin\frac{2\pi x}{a}\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right)-\frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}+\frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}d\left(\frac{x}{a}\right)$$

$$\left.+\frac{1}{2}\int_0^1\left\{-\frac{\lambda}{2}\left(\frac{\pi x}{a}\right)-\frac{\lambda^3}{1+2\lambda^2}\left(\frac{\pi x}{a}\right)\cos\frac{\pi x}{a}+\frac{a_1}{2}\sqrt{2}\lambda^2\sinh\frac{\sqrt{2}\lambda\pi x}{a}\right\}^2d\left(\frac{x}{a}\right)\right]$$

$$\frac{1}{E^2 ab}\int_0^a\int_0^b \tau_{xy}^2\,dx\,dy = \left(\frac{a}{R}\right)^4\left(\frac{f}{8}\right)^2\Bigg[\frac{\lambda^2(1+\lambda^2)^2}{4(1+2\lambda^2)^4} + \frac{\lambda^4(1+\lambda^2)}{16(1+2\lambda^2)^3}(f\pi^2) + \frac{\lambda^6}{256(1+2\lambda^2)^2}(f\pi^2)^2$$

$$+ \frac{\lambda^6}{4096(2+\lambda^2)^2}(f\pi^2)^2 + \frac{\lambda^6}{256(1+8\lambda^2)^2}(f\pi^2)^2 + \frac{\lambda^6}{16384(1+2\lambda^2)^2}(f\pi^2)^2$$

$$+\left\{\frac{\lambda(1+\lambda^2)}{(1+2\lambda^2)^2}+\frac{\lambda^3}{1+2\lambda^2}\frac{f\pi^2}{8}\right\}\left\{-\frac{\lambda}{2}+\frac{1}{4}\frac{\lambda^3}{1+\lambda^2}+\frac{1}{2}\frac{\lambda}{(1+2\lambda^2)^2}\right\}$$

$$+\frac{\lambda^3}{32(2+\lambda^2)}(f\pi^2)\left\{+\frac{\lambda}{4}-\frac{2}{3}\frac{\lambda^3}{1+2\lambda^2}-\frac{1}{2}\frac{\lambda}{(1+2\lambda^2)(2+\lambda^2)}\right\}$$

$$+\frac{1}{2}\Bigg\{\frac{\lambda^2}{4}\frac{\pi^2}{3}-\frac{2\lambda^4}{1+2\lambda^2}-\frac{1}{4}\frac{1}{(1+2\lambda^2)}\frac{1}{\sinh\sqrt{2}\lambda\pi}\left(\sqrt{2}\lambda\pi\cosh\sqrt{2}\lambda\pi-\sinh\sqrt{2}\lambda\pi\right)$$

$$+\frac{\lambda^6}{(1+2\lambda^2)^2}\left(\frac{\pi^2}{6}+\frac{1}{4}\right)-\frac{\lambda^4\pi}{(1+2\lambda^2)^2\sinh\sqrt{2}\lambda\pi}\left(-\frac{\sqrt{2}\lambda\cosh\sqrt{2}\lambda\pi}{1+2\lambda^2}+\frac{1}{\pi}\frac{(2\lambda^2-1)\sinh\sqrt{2}\lambda\pi}{(1+2\lambda^2)^2}\right)$$

$$+\frac{\pi^2\lambda^2}{4(1+2\lambda^2)^2\sinh^2\sqrt{2}\lambda\pi}\left(\frac{\sinh 2\sqrt{2}\lambda\pi}{4\sqrt{2}\lambda\pi}-\frac{1}{2}\right)\Bigg\}\Bigg]$$

$$\frac{E_1}{Eabt} = [\text{illegible}]\left[H_1(\lambda)(f\pi^2)^2 - H_2(\lambda)(f\pi^2) + H_3(\lambda)\right] + \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$H_1(\lambda) = \frac{105}{32768} + \frac{17\lambda^4}{16384} + \frac{\lambda^4}{2048(1+8\lambda^2)^2} + \frac{\lambda^4}{32768(1+2\lambda^2)^2} + \frac{\lambda^4}{512(1+2\lambda^2)^2} + \frac{\lambda^4}{2048(2+\lambda^2)^2}$$

$$+ \frac{\lambda^6}{256(1+2\lambda^2)^2} + \frac{\lambda^6}{4096(2+\lambda^2)^2} + \frac{\lambda^6}{256(1+8\lambda^2)^2} + \frac{\lambda^6}{16384(1+2\lambda^2)^2}$$

$$= \frac{105}{32768} + \frac{17\lambda^4}{16384} + \frac{\lambda^4}{2048(1+8\lambda^2)} + \frac{\lambda^4}{32768(1+2\lambda^2)} + \frac{\lambda^4}{512(1+2\lambda^2)} + \frac{\lambda^4}{4096(2+\lambda^2)}$$

$$\boxed{H_1 = \frac{105}{32768} + \frac{17\lambda^4}{16384} + \frac{\lambda^4}{2048(1+8\lambda^2)} + \frac{65}{32768}\frac{\lambda^4}{(1+2\lambda^2)} + \frac{1}{4096}\frac{\lambda^4}{(2+\lambda^2)}}$$

480

$$H_2(\lambda) = \frac{5}{48} + \frac{1}{32}\frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{64}\frac{\lambda^4}{(1+2\lambda^2)^2} + \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{96}\frac{\lambda^4}{(2+\lambda^2)(1+2\lambda^2)}$$

$$-\frac{1}{32}\frac{\lambda^4}{(1+2\lambda^2)(2+\lambda^2)^2} \quad \frac{1}{16}\frac{\lambda^4(1+\lambda^2)}{(1+2\lambda^2)^3} + \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)} - \frac{1}{32}\frac{\lambda^6}{(1+2\lambda^2)^2} - \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)^3}$$

$$-\frac{1}{128}\frac{\lambda^4}{(2+\lambda^2)} + \frac{1}{48}\frac{\lambda^6}{(2+\lambda^2)(1+2\lambda^2)} + \frac{1}{64}\frac{\lambda^4}{(1+2\lambda^2)(2+\lambda^2)^2}$$

$$\boxed{H_2 = \frac{5}{48} + \frac{3}{64}\frac{\lambda^4}{(1+2\lambda^2)} + \frac{\lambda^4}{384(2+\lambda^2)} - \frac{1}{64}\frac{\lambda^4}{(1+2\lambda^2)(2+\lambda^2)^2}}$$

$$H_3(\lambda) = \frac{\pi^2}{8} - \frac{3}{16}\Big] + \frac{1}{16} + \frac{1}{8}\frac{\lambda^4}{(1+2\lambda^2)^4} - \frac{1}{4}\frac{\lambda^2}{(1+2\lambda^2)} - \frac{1}{8}\frac{1}{(1+2\lambda^2)} + \frac{1}{8}\frac{\lambda^4}{(1+2\lambda^2)^3} + \frac{1}{2}\frac{\lambda^4}{(1+2\lambda^2)^4}$$

$$+\frac{1}{4}\left(\frac{\pi^2}{6} - \frac{1}{4}\right)\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2 + \frac{1}{4}\frac{\lambda^2}{(1+2\lambda^2)^3}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} - \frac{\lambda^4}{(1+2\lambda^2)^4}$$

$$+\frac{1}{16(1+2\lambda^2)^2}\frac{\frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)^2}\Bigg] + \frac{\lambda^2(1+\lambda^2)^2}{4(1+2\lambda^2)^4} - \frac{1}{2}\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{1}{4}\frac{\lambda^4(1+\lambda^2)}{(1+2\lambda^2)^3}$$

$$+\frac{1}{2}\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^4} + \frac{\lambda^2}{8}\frac{\pi^2}{3} - \frac{\lambda^4}{(1+2\lambda^2)} - \frac{1}{8}\frac{1}{(1+2\lambda^2)}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{1}{8}\frac{1}{(1+2\lambda^2)} + \frac{\lambda^6}{2(1+2\lambda^2)^2}\left(\frac{\pi^2}{6} + \frac{1}{4}\right)$$

$$+\frac{1}{2}\frac{\lambda^4}{(1+2\lambda^2)^3}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{1}{2}\frac{\lambda^4(1-2\lambda^2)}{(1+2\lambda^2)^4} + \frac{1}{16(1+2\lambda^2)^2}\frac{\frac{1}{2} + \frac{1}{4\sqrt{2}\lambda\pi}\sinh 2\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)^2}$$

488

$$H_3(\lambda) = -\frac{1}{16(1+2\lambda^2)^2}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8} - \frac{1}{8} + \frac{1}{4}\left(\frac{\pi^2}{6} - \frac{1}{4}\right)\left(\frac{\lambda^2}{1+2\lambda^2}\right)^2 + \frac{1}{2}\left(\frac{\pi^2}{6}+\frac{1}{4}\right)\frac{\lambda^6}{(1+2\lambda^2)^2}$$

$$+ \frac{3}{8}\frac{\lambda^2(2-\lambda^2)}{(1+2\lambda^2)^3} - \frac{1}{8} + \frac{1}{8}\frac{\lambda^4(3+2\lambda^2)}{(1+2\lambda^2)^3} - \frac{1}{2}\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{\lambda^2}{24}\pi^2 + \frac{1}{8}\frac{(1-8\lambda^4)}{(1+2\lambda^2)}$$

$$= -\frac{1}{16(1+2\lambda^2)^2}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8} - \frac{1}{4} + \frac{\pi^2}{24}\frac{\lambda^4}{(1+2\lambda^2)} + \frac{\pi^2}{24}\lambda^2 - \frac{1}{16}\frac{\lambda^4(1-2\lambda^2)}{(1+2\lambda^2)^2}$$

$$+ \frac{1}{4}\frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3} - \frac{1}{2}\frac{\lambda^2(1\,1\lambda^2)}{(1+2\lambda^2)^2} + \frac{1}{8} - \frac{1}{4}\frac{\lambda^2(1+4\lambda^2)}{(1+2\lambda^2)}$$

$$= -\frac{1}{16(1+2\lambda^2)^2}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{8}\left\{1 + \frac{1}{3}\frac{\lambda^2(1+3\lambda^2)}{1+2\lambda^2}\right\} - \frac{1}{8} - \frac{1}{16}\frac{\lambda^4}{(1+2\lambda^2)}$$

$$+ \frac{1}{4}\frac{\lambda^6}{(1+2\lambda^2)^2} - \frac{1}{2}\frac{\lambda^2(1+\lambda^2)}{(1+2\lambda^2)^2} + \frac{1}{4}\frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3} - \frac{1}{4}\frac{\lambda^2(1+4\lambda^2)}{(1+2\lambda^2)}$$

$$H_3 = \frac{1}{8}(\pi^2-1) - \frac{1}{16(1+2\lambda^2)^2}\frac{\cosh\sqrt{2}\lambda\pi}{\left(\frac{\sinh\sqrt{2}\lambda\pi}{\sqrt{2}\lambda\pi}\right)} + \frac{\pi^2}{24}\frac{\lambda^2(1+3\lambda^2)}{(1+2\lambda^2)} - \frac{1}{16}\frac{\lambda^2(4+17\lambda^2)}{(1+2\lambda^2)} + \frac{\lambda^2(\lambda^4-2\lambda^2-2)}{4(1+2\lambda^2)^2}$$

$$+ \frac{1}{4}\frac{\lambda^2(3+\lambda^4)}{(1+2\lambda^2)^3}$$

49e

$$\frac{P}{abtE} = \left(\frac{a}{R}\right)^4 \left(\frac{f}{\delta}\right)^2 \left[H_1 \left(\frac{1}{\pi^2}\right)^2 - H_2 \left(\frac{1}{\pi^2}\right) + H_3 \right] + \left(\frac{t}{R}\right)^2 \left(\frac{f}{\delta}\right)^2 \pi^4 \left[\frac{1}{32}(1+\lambda^4) + \frac{1}{48}\lambda^2 \right]$$

$$- \left(\frac{a}{R}\right)^2 \left(\frac{f}{\delta}\right)^2 \left(\frac{3}{8}\lambda^2\pi^2\right) \frac{\sigma}{E} - \frac{1}{2}\left(\frac{\sigma}{E}\right)^2$$

$$\frac{3}{4}\lambda^2\pi^2 \frac{\sigma}{E} = \left(\frac{a}{R}\right)^2 \left[4H_1 \left(\frac{1}{\pi^2}\right)^2 - 3H_2 \left(\frac{1}{\pi^2}\right) + 2H_3 \right] + \left(\frac{t}{R}\right)^2 \pi^4 \left[\frac{1}{16}(1+\lambda^4) + \frac{1}{24}\lambda^2 \right]$$

$$\lambda^2 K = \gamma^2 \left[\frac{16}{3} H_1 f^2 \pi^2 - 4 H_2 f + \frac{8}{3} H_3 \frac{1}{\pi^2} \right] + \frac{\pi^2}{\gamma^2} \left[\frac{1}{12}(1+\lambda^4) + \frac{1}{18}\lambda^2 \right]$$

$$= \frac{\pi^2}{\gamma^2} \left[\frac{64}{3} H_1 \left(\frac{\delta}{t}\right)^2 + \left\{ \frac{1}{12}(1+\lambda^4) + \frac{1}{18}\lambda^2 \right\} \right] + \frac{8}{3}\frac{\gamma^2}{\pi^2} H_3 - 8H_2\left(\frac{\delta}{t}\right)$$

$$= 2 \left[\frac{512}{9} H_1 H_3 \left(\frac{\delta}{t}\right)^2 + H_3 \left\{ \frac{2}{9}(1+\lambda^4) + \frac{4}{27}\lambda^2 \right\} \right]^{\frac{1}{2}} - 8H_2\left(\frac{\delta}{t}\right)$$

483

$\lambda = 1.$ <u>484</u>

$$\frac{512}{9} H_1 = \frac{\cancel{105}^{35}}{64 \times \cancel{9}_3} + \frac{17}{32 \times 9} + \frac{1}{4 \times 81} + \frac{65}{64 \times 27} + \frac{1}{8 \times 27}$$

$$= 0.1822917 + 0.0590278 + 0.0030864 + 0.0376157 + 0.0046296$$

$$= 0.2866512$$

$$8 H_2 = \frac{40}{48} + \frac{3}{8}\frac{1}{3} + \frac{1}{48}\frac{1}{3} - \frac{1}{8}\frac{1}{27}$$

$$= 0.8333333 + 0.1250000 + 0.00694444 - 0.0046296$$

$$= 0.9606481$$

$$H_3 = 1.1087006 - \frac{4.444113}{16 \times 9} + \frac{\pi^2}{\cancel{24}_6}\frac{\cancel{4}}{3} - \frac{1}{16}\frac{21}{3} - \frac{1}{4}\frac{3}{9} + \frac{1}{27}$$

$$= 1.1087006 - 0.0308619 + 0.5498311 - 0.4375 - 0.0833333 + 0.0370370$$

$$= 1.1438735$$

$$K = 2\left[0.3278927\left(\frac{\delta}{t}\right)^2 + 0.6778509\right]^{\frac{1}{2}} - 0.9606481\left(\frac{\delta}{t}\right)$$

$$0.6557854\left(\frac{\delta}{t}\right) = 0.9606481\left[0.3278927\left(\frac{\delta}{t}\right)^2 + 0.6778509\right]^{\frac{1}{2}}$$

$$0.4300545\left(\frac{\delta}{t}\right)^2 = 0.6255512$$

$$\frac{0.3025940}{0.1274505}$$

$$\left(\frac{\delta}{t}\right)^2 = 4.908189 \qquad \left(\frac{\delta}{t}\right) = 2.215444$$

$$K_{min} = 0.8965 \qquad K_0 = 1.646636$$

$\lambda = 0.5$ 485

$$\frac{512}{9} H_1 = 0.1822917 + \frac{17 \times 0.0625}{288} + \frac{0.0625}{108} + \frac{65}{} \frac{0.0625}{864} + \frac{1}{} \frac{0.0625}{162}$$

$$= 0.1822917 + 0.0625 (0.05932777 + 0.00925926 + 0.07523148 + 0.00617284)$$

$$= 0.1822917 + 0.0093557 = 0.1916474 \qquad 0.7665896$$

$$8 H_2 = 0.8333333 + 0.0625 \left(\frac{3}{8} \frac{1}{1.5} + \frac{1}{48} \frac{1}{2.25} - \frac{1}{8} \frac{1}{1.5 \times 2.25^2} \right)$$

$$= 0.8333333 + 0.0625 (0.25 + 0.00925926 - 0.01646091)$$

$$= 0.8485082 \qquad 3.3940328$$

$$H_3 = 1.1087006 - \frac{1}{36} 2.274322 + \pi^2 \frac{0.25 \times 1.75}{36} - \frac{1}{} \frac{0.25 \times 8.25}{24} - \frac{0.25 \times 2.4375}{9} + \frac{0.25 \times 3.0625}{13.5}$$

$$= 1.1087006 - 0.06317511 + 0.1199431 - 0.08593750 - 0.06770833 + 0.05671296$$

$$= 1.1476353 \qquad 4.5905412$$

$$K = 2 \left[3.5190611 \left(\frac{f}{E}\right)^2 + 5.0155909 \right]^{\frac{1}{2}} - 3.3940328 \left(\frac{f}{E}\right)$$

$$49.5351641 \left(\frac{f}{E}\right)^2 = 57.7762917 \qquad \left(\frac{f}{E}\right) = 2.534056$$

$$40.5376787 \qquad 6.421448$$

$$8.9974854$$

$$K_{min} = 1.4 - -$$

［Ｇｅｎｅｒａｌ Ｉｎｆｏｒｍａｔｉｏｎ］
书名＝钱学森力学手稿　　４　英文
作者＝钱学森著
页数＝１４９
ＳＳ号＝１３２４６０１１
出版日期＝２０１２．０３